

Quantum phase transitions in cascading gauge theory

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Abstract

We study a ground state of $\mathcal{N} = 1$ supersymmetric $SU(K + P) \times SU(K)$ cascading gauge theory of Klebanov et.al [1, 2] on $R \times S^3$ at zero temperature. A radius of S^3 sets a compactification scale μ . An interplay between μ and the strong coupling scale Λ of the theory leads to an interesting pattern of quantum phases of the system. For $\mu \geq \mu_{\chi\text{SB}} = 1.240467(8)\Lambda$ the ground state of the theory is chirally symmetric. At $\mu = \mu_{\chi\text{SB}}$ the theory undergoes the first-order transition to a phase with spontaneous breaking of the chiral symmetry. We further demonstrate that the chirally symmetric ground state of cascading gauge theory becomes perturbatively unstable at scales below $\mu_c = 0.950634(5)\mu_{\chi\text{SB}}$. Finally, we point out that for $\mu < 1.486402(5)\Lambda$ the stress-energy tensor of cascading gauge theory can source inflation of a closed Universe.

August 30, 2011

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1 Introduction and Summary

Consider $\mathcal{N} = 1$ four-dimensional supersymmetric $SU(K + P) \times SU(K)$ gauge theory with two chiral superfields A_1, A_2 in the $(K + P, \overline{K})$ representation, and two fields B_1, B_2 in the $(\overline{K + P}, K)$ in Minkowski space-time. Perturbatively, this gauge theory has two gauge couplings g_1, g_2 associated with two gauge group factors, and a quartic superpotential

$$W \sim \text{Tr} (A_i B_j A_k B_\ell) \epsilon^{ik} \epsilon^{j\ell}. \quad (1.1)$$

When $P = 0$ above theory flows in the infrared to a superconformal fixed point, commonly referred to as Klebanov-Witten (KW) theory [3]. At the IR fixed point KW gauge theory is strongly coupled — the superconformal symmetry together with $SU(2) \times SU(2) \times U(1)$ global symmetry of the theory implies that anomalous dimensions of chiral superfields $\gamma(A_i) = \gamma(B_i) = -\frac{1}{4}$, *i.e.*, non-perturbatively large.

When $P \neq 0$, conformal invariance of the above $SU(K + P) \times SU(K)$ gauge theory is broken. It is useful to consider an effective description of this theory at energy scale μ with perturbative couplings $g_i(\mu) \ll 1$. It is straightforward to evaluate NSVZ beta-functions for the gauge couplings. One finds that while the sum of the gauge couplings does not run

$$\frac{d}{d \ln \mu} \left(\frac{\pi}{g_s} \equiv \frac{4\pi}{g_1^2(\mu)} + \frac{4\pi}{g_2^2(\mu)} \right) = 0, \quad (1.2)$$

the difference between the two couplings is

$$\frac{4\pi}{g_2^2(\mu)} - \frac{4\pi}{g_1^2(\mu)} \sim P [3 + 2(1 - \gamma_{ij})] \ln \frac{\mu}{\Lambda}, \quad (1.3)$$

where Λ is the strong coupling scale of the theory and γ_{ij} is an anomalous dimension of operators $\text{Tr} A_i B_j$. Given (1.3) and (1.2) it is clear that the effective weakly coupled description of $SU(K + P) \times SU(K)$ gauge theory can be valid only in a finite-width energy band centered about μ scale. Indeed, extending effective description both to the UV and to the IR one necessarily encounters strong coupling in one or the other gauge group factor. As explained in [2], to extend the theory past the strongly coupled region(s) one must perform a Seiberg duality [4]. Turns out, in this gauge theory, a

Seiberg duality transformation is a self-similarity transformation of the effective description so that $K \rightarrow K - P$ as one flows to the IR, or $K \rightarrow K + P$ as the energy increases. Thus, extension of the effective $SU(K + P) \times SU(K)$ description to all energy scales involves an infinite sequence - a *cascade* - of Seiberg dualities where the rank of the gauge group is not constant along RG flow, but changes with energy according to [5–7]

$$K = K(\mu) \sim 2P^2 \ln \frac{\mu}{\Lambda}, \quad (1.4)$$

at least as $\mu \gg \Lambda$. To see (1.4), note that the rank changes by $\Delta K \sim P$ as $P\Delta \left(\ln \frac{\mu}{\Lambda}\right) \sim 1$. Although there are infinitely many duality cascade steps in the UV, there is only a finite number of duality transformations as one flows to the IR (from a given scale μ). The space of vacua of a generic cascading gauge theory was studied in details in [8]. In the simplest case, when $K(\mu)$ is an integer multiple of P , cascading gauge theory confines in the infrared with a spontaneous breaking of the chiral symmetry $U(1) \supset \mathbb{Z}_2$ [2]. Here, the full global symmetry of the ground state is $SU(2) \times SU(2) \times \mathbb{Z}_2$.

Effective description of cascading gauge theory in the UV suggests that it must be ultimately defined as a theory with an infinite number of degrees of freedom. If so, an immediate concern is whether such a theory is renormalizable as a four dimensional quantum field theory, *i.e.*, whether a definite prescription can be made for the computation of all gauge invariant correlation functions in the theory. As was pointed out in [2], whenever $g_s K(\mu) \gg 1$, cascading gauge theory allows for a dual holographic description [9, 10] as type IIB supergravity on a warped deformed conifold with fluxes. The duality is always valid in the UV of cascading gauge theory; if, in addition, $g_s P \gg 1$ the holographic correspondence is valid in the IR as well. It was shown in [11] that a cascading gauge theory *defined* by its holographic dual as an RG flow of type IIB supergravity on a warped deformed conifold with fluxes is holographically renormalizable as a four dimensional quantum field theory.

In this paper¹ we study the properties of the ground state of strongly coupled cascading gauge theory on $R \times S^3$ at zero temperature². The radius f_0 of the S^3 sets a compactification scale

$$\mu \equiv \frac{1}{f_0}. \quad (1.5)$$

In the limit $\frac{\mu}{\Lambda} \rightarrow 0$ the ground state of the theory has a (spontaneously) broken chiral

¹See [12] for a related early work.

²Thermodynamics and the hydrodynamic transport of cascading gauge theory plasma are discussed in [5, 13–20].

symmetry [2], while for $\frac{\mu}{\Lambda} \gg 1$ the chiral symmetry of cascading gauge theory is expected to be restored [12]. Thus, one expects that there is a critical scale $\mu_{\chi\text{SB}} \sim \Lambda$ above which the ground state of the theory is chirally symmetric, while at $\mu < \mu_{\chi\text{SB}}$ the chiral symmetry of the ground state is spontaneously broken. We explicitly confirm a quantum phase transition (QPT) of this type in cascading gauge theory. Specifically, we compute the difference of the ground state energy densities in the symmetric \mathcal{E}^s and in the broken \mathcal{E}^b phases and find that

$$\Delta\mathcal{E}\left(\frac{\mu}{\Lambda}\right) \equiv \mathcal{E}^b - \mathcal{E}^s \propto +(\mu - \mu_{\chi\text{SB}}), \quad \frac{|\mu - \mu_{\chi\text{SB}}|}{\Lambda} \ll 1. \quad (1.6)$$

Since

$$\left. \frac{d\Delta\mathcal{E}}{d\ln\mu} \right|_{\mu=\mu_{\chi\text{SB}}} \neq 0, \quad (1.7)$$

the chiral symmetry breaking QPT in cascading gauge theory is of the first-order. We further study in details the spectrum of linearized chiral symmetry breaking (χSB) fluctuations in the symmetric phase of cascading gauge theory and identify modes which become tachyonic for

$$\mu < \mu_c = 0.950634(5) \mu_{\chi\text{SB}}. \quad (1.8)$$

The rest of the paper is organized as follows. In the next section we recall the gravitational low-energy effective action realizing holographic dual to strongly coupled cascading gauge theory, and the effective action of the χSB fluctuations about a chirally symmetric state of the theory [17]. In section 3 we present equations of motion and the asymptotics of the gravitational dual to a chirally symmetric ground state of cascading gauge theory. We discuss various scaling symmetries of the relevant gravitational solution, explain the encoding of the physical parameters of cascading gauge theory in their dual gravitational description, outline the numerical procedure for obtaining the gravitational solution, and compute the energy density \mathcal{E}^s and the pressure \mathcal{P}^s of this ground state as a function of³ $\frac{\mu}{\Lambda}$. To leading order in $\delta \equiv \left(\ln \frac{\mu^2}{\Lambda^2 P^2 g_0}\right)^{-1}$ we find

$$\begin{aligned} \mathcal{E}^s &= \frac{\mu^4}{8\pi G_5} \frac{1}{32} \left(\frac{P^2 g_0}{\delta} + P^2 g_0 \ln \frac{P^2 g_0}{\delta} \right)^2 \left(1 - 2.272588(7) \delta + \mathcal{O}(\delta^2) \right), \\ \mathcal{P}^s &= \frac{\mu^4}{8\pi G_5} \frac{1}{96} \left(\frac{P^2 g_0}{\delta} + P^2 g_0 \ln \frac{P^2 g_0}{\delta} \right)^2 \left(1 + 1.727411(3) \delta + \mathcal{O}(\delta^2) \right), \end{aligned} \quad (1.9)$$

³See (3.60) for a precise definition of Λ .

where g_0 is the asymptotic values of the string coupling (see (3.17)). Rather interestingly, we find that the energy density \mathcal{E}^s of cascading gauge theory chirally symmetric phase is negative for $\mu < 2.010798(8)\Lambda$, while the pressure \mathcal{P}^s becomes negative for somewhat smaller compactification scales $\mu < 1.375284(4)\Lambda$. For $\mu < 1.486402(5)\Lambda$ the combination $(\mathcal{E}^s + 3\mathcal{P}^s)$ becomes negative, which implies that cascading gauge theory compactified on sufficiently small S^3 would lead to an inflating closed Universe when coupled to four-dimensional Einstein gravity⁴. In section 4 we study the spectrum of linearized χ SB fluctuations about a chirally symmetric ground state of cascading gauge theory on S^3 . A mass of a generic χ SB state in the spectrum depends on the S^3 eigenvalue L of its wavefunction, and an integer q which quantizes its radial wavefunction. For each pair $\{L, q\}$ there are two branches in the spectrum associated with the non-analytic dependence of the mass on $\sqrt{\delta}$. This is evident from the (semi-)analytic analysis of the spectrum in the limit $\delta \rightarrow 0$. Specifically, we find that the mass-squared ω^2 of $\{L = 0, q = 1\}$ states is given by

$$\frac{\omega^2}{\mu^2} = 9 \mp 6\sqrt{2} \sqrt{\delta} + 0.077172(8) \delta + \mathcal{O}(\delta^{3/2}). \quad (1.10)$$

The lighter of the two states in (1.10) eventually becomes tachyonic as δ (or equivalently $\frac{\Lambda}{\mu}$) becomes sufficiently large:

$$\left. \frac{\omega^2}{\mu^2} \right|_{L=0, q=1} < 0 \quad \text{if} \quad \mu < \mu_c = 1.179231(5) \Lambda. \quad (1.11)$$

An interesting question is whether or not the final state associated with the condensation of χ SB tachyons below μ_c can be continuously connected to a chirally symmetric ground state in the limit $\mu \rightarrow \mu_c$ (from below). To address it, we mass-deform the cascading gauge theory at $\mu = \mu_* < \mu_c$, thus explicitly breaking the chiral symmetry. We show that the χ SB condensates from the explicit breaking vanish as the mass-deformation parameter vanishes⁵. In section 5 we construct a new state of cascading gauge theory on S^3 with spontaneously broken chiral symmetry. We begin with the supersymmetric state of cascading gauge theory on $R^{3,1}$ with χ SB [2] and “continuously compactify” R^3 to S^3 (see section 5.1 for details). During the “compactification”

⁴Expanding S^3 would ultimately end cascading gauge theory driven inflation. Further cosmological aspects of cascading gauge theory will be discussed elsewhere.

⁵This analysis are equivalent to the one in [17] where it was shown that χ SB tachyons of cascading gauge theory plasma do not condense to a homogeneous and isotropic state continuously connected to a chirally symmetric state of the plasma.

process the chiral symmetry of the gauge theory remains spontaneously broken. Next we compare the energies of the chirally symmetric state of section 3, \mathcal{E}^s , and that of the newly constructed state, \mathcal{E}^b , as a function of the compactification scale μ . We show that the new state with spontaneously broken chiral symmetry is energetically favourable, *i.e.*, $\mathcal{E}^s > \mathcal{E}^b$, for

$$\mu < \mu_{\chi\text{SB}}, \quad \mu_{\chi\text{SB}} = 1.240467(8)\Lambda > \mu_c. \quad (1.12)$$

This quantum phase transition is of the first order, see (1.7). Notice that this transition occurs *prior to* (at higher compactification scales than) the tachyon condensation in the chirally symmetric phase. One possibility⁶ is that the tachyon discussed in section 4.3 condenses into this new phase, which would also explain the absence of the χSB phase of cascading gauge theory continuously connected to a chirally symmetric phase as $\mu \rightarrow \mu_c$. Another possibility⁷ is that the end point of the tachyon condensation would produce a, yet undiscovered, state which is not $SO(4)$ -invariant. Indeed, while we established the condensation of $SO(4)$ -invariant ($L = 0$) state in the spectrum of χSB fluctuations, from figure 5 is it likely that the $SO(4)$ non-invariant state ($L = 3$) would condense as well, albeit at scales lower than μ_c . The latter can also explain while an $SO(4)$ non-invariant state with χSB (if it exists) is not continuously connected to the $SO(4)$ -invariant chirally symmetric state at $\mu = \mu_c$. We hope to resolve these issues in the future work.

2 Dual effective actions of cascading gauge theory

Consider $SU(2) \times SU(2) \times \mathbb{Z}_2$ invariant states of cascading gauge theory on a 4-dimensional manifold $\mathcal{M}_4 \equiv \partial\mathcal{M}_5$. Effective gravitational action on a 5-dimensional

⁶In the context of cascading gauge theory plasma [17] this scenario was advocated by Ofer Aharony [21].

⁷In the context of cascading gauge theory plasma this scenario was advocated in [17].

manifold \mathcal{M}_5 describing holographic dual of such states was derived in [17]:

$$\begin{aligned}
S_5 [g_{\mu\nu}, \Omega_i, h_i, \Phi] = & \frac{108}{16\pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \Omega_1 \Omega_2^2 \Omega_3^2 \left\{ R_{10} - \frac{1}{2} (\nabla \Phi)^2 \right. \\
& - \frac{1}{2} e^{-\Phi} \left(\frac{(h_1 - h_3)^2}{2\Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{1}{\Omega_3^4} (\nabla h_1)^2 + \frac{1}{\Omega_2^4} (\nabla h_3)^2 \right) \\
& - \frac{1}{2} e^{\Phi} \left(\frac{2}{\Omega_2^2 \Omega_3^2} (\nabla h_2)^2 + \frac{1}{\Omega_1^2 \Omega_2^4} \left(h_2 - \frac{P}{9} \right)^2 + \frac{1}{\Omega_1^2 \Omega_3^4} h_2^2 \right) \\
& \left. - \frac{1}{2\Omega_1^2 \Omega_2^4 \Omega_3^4} \left(4\Omega_0 + h_2 (h_3 - h_1) + \frac{1}{9} P h_1 \right)^2 \right\}, \tag{2.1}
\end{aligned}$$

where Ω_0 is a constant, R_{10} is given by

$$\begin{aligned}
R_{10} = & R_5 + \left(\frac{1}{2\Omega_1^2} + \frac{2}{\Omega_2^2} + \frac{2}{\Omega_3^2} - \frac{\Omega_2^2}{4\Omega_1^2 \Omega_3^2} - \frac{\Omega_3^2}{4\Omega_1^2 \Omega_2^2} - \frac{\Omega_1^2}{\Omega_2^2 \Omega_3^2} \right) - 2\Box \ln (\Omega_1 \Omega_2^2 \Omega_3^2) \\
& - \left\{ (\nabla \ln \Omega_1)^2 + 2 (\nabla \ln \Omega_2)^2 + 2 (\nabla \ln \Omega_3)^2 + (\nabla \ln (\Omega_1 \Omega_2^2 \Omega_3^2))^2 \right\}, \tag{2.2}
\end{aligned}$$

and R_5 is the five dimensional Ricci scalar of the metric

$$ds_5^2 = g_{\mu\nu}(y) dy^\mu dy^\nu. \tag{2.3}$$

All the covariant derivatives ∇_λ are with respect to the metric (2.3). Finally, G_5 is the five dimensional effective gravitational constant

$$G_5 \equiv \frac{729}{4\pi^3} G_{10}, \tag{2.4}$$

where G_{10} is a 10-dimensional gravitational constant of type IIB supergravity.

From (2.1) we obtain the following equations of motion:

$$\begin{aligned}
0 = & \Box \Phi + \nabla \Phi \nabla \ln \Omega_1 \Omega_2^2 \Omega_3^2 + \frac{1}{2} e^{-\Phi} \left(\frac{(h_1 - h_3)^2}{2\Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{1}{\Omega_3^4} (\nabla h_1)^2 + \frac{1}{\Omega_2^4} (\nabla h_3)^2 \right) \\
& - \frac{1}{2} e^{\Phi} \left(\frac{2}{\Omega_2^2 \Omega_3^2} (\nabla h_2)^2 + \frac{1}{\Omega_1^2 \Omega_2^4} \left(h_2 - \frac{P}{9} \right)^2 + \frac{1}{\Omega_1^2 \Omega_3^4} h_2^2 \right), \tag{2.5}
\end{aligned}$$

$$\begin{aligned}
0 = & \Box h_1 - \nabla \Phi \nabla h_1 + \nabla h_1 \nabla \ln \frac{\Omega_1 \Omega_2^2}{\Omega_3^2} + \frac{(h_3 - h_1) \Omega_3^2}{2\Omega_1^2 \Omega_2^2} + \frac{(h_2 - \frac{P}{9}) e^{\Phi}}{\Omega_1^2 \Omega_2^4} \left(4\Omega_0 \right. \\
& \left. + h_2 (h_3 - h_1) + \frac{1}{9} P h_1 \right), \tag{2.6}
\end{aligned}$$

$$\begin{aligned}
0 = & \square h_2 + \nabla \Phi \nabla h_2 + \nabla h_2 \nabla \ln \Omega_1 - \frac{h_2 \Omega_2^2}{2\Omega_1^2 \Omega_3^2} - \frac{(h_2 - \frac{P}{9}) \Omega_3^2}{2\Omega_1^2 \Omega_2^2} + \frac{(h_1 - h_3) e^{-\Phi}}{2\Omega_1^2 \Omega_2^2 \Omega_3^2} \left(4\Omega_0 \right. \\
& \left. + h_2 (h_3 - h_1) + \frac{1}{9} P h_1 \right), \tag{2.7}
\end{aligned}$$

$$\begin{aligned}
0 = & \square h_3 - \nabla \Phi \nabla h_3 + \nabla h_3 \nabla \ln \frac{\Omega_1 \Omega_3^2}{\Omega_2^2} + \frac{(h_1 - h_3) \Omega_2^2}{2\Omega_1^2 \Omega_3^2} - \frac{h_2 e^\Phi}{\Omega_1^2 \Omega_3^4} \left(4\Omega_0 + h_2 (h_3 - h_1) \right. \\
& \left. + \frac{1}{9} P h_1 \right), \tag{2.8}
\end{aligned}$$

$$\begin{aligned}
0 = & \Omega_1^{-1} \square \Omega_1 + \nabla \ln \Omega_1 \nabla \ln \Omega_2^2 \Omega_3^2 + \frac{(\Omega_2^2 - \Omega_3^2)^2 - 4\Omega_1^4}{4\Omega_1^2 \Omega_2^2 \Omega_3^2} - \frac{e^{-\Phi}}{8\Omega_3^4} (\nabla h_1)^2 - \frac{e^\Phi}{4\Omega_2^2 \Omega_3^2} (\nabla h_2)^2 \\
& - \frac{e^{-\Phi}}{8\Omega_2^4} (\nabla h_3)^2 + \frac{3(h_1 - h_3)^2 e^{-\Phi}}{16\Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{3h_2^2 e^\Phi}{8\Omega_1^2 \Omega_3^4} + \frac{3(h_2 - \frac{P}{9})^2 e^\Phi}{8\Omega_1^2 \Omega_2^4} + \frac{1}{4\Omega_1^2 \Omega_2^4 \Omega_3^4} \left(4\Omega_0 \right. \\
& \left. + h_2 (h_3 - h_1) + \frac{1}{9} P h_1 \right)^2, \tag{2.9}
\end{aligned}$$

$$\begin{aligned}
0 = & \Omega_2^{-1} \square \Omega_2 + \nabla \ln \Omega_2 \nabla \ln \Omega_1 \Omega_2 \Omega_3^2 + \frac{(2\Omega_1^2 - \Omega_3^2)^2 - 4\Omega_1^2 \Omega_3^2 - \Omega_2^4}{8\Omega_1^2 \Omega_2^2 \Omega_3^2} - \frac{e^{-\Phi}}{8\Omega_3^4} (\nabla h_1)^2 \\
& + \frac{e^\Phi}{4\Omega_2^2 \Omega_3^2} (\nabla h_2)^2 + \frac{3e^{-\Phi}}{8\Omega_2^4} (\nabla h_3)^2 + \frac{(h_1 - h_3)^2 e^{-\Phi}}{16\Omega_1^2 \Omega_2^2 \Omega_3^2} - \frac{h_2^2 e^\Phi}{8\Omega_1^2 \Omega_3^4} + \frac{3(h_2 - \frac{P}{9})^2 e^\Phi}{8\Omega_1^2 \Omega_2^4} \\
& + \frac{1}{4\Omega_1^2 \Omega_2^4 \Omega_3^4} \left(4\Omega_0 + h_2 (h_3 - h_1) + \frac{1}{9} P h_1 \right)^2, \tag{2.10}
\end{aligned}$$

$$\begin{aligned}
0 = & \Omega_3^{-1} \square \Omega_3 + \nabla \ln \Omega_3 \nabla \ln \Omega_1 \Omega_2^2 \Omega_3 + \frac{(2\Omega_1^2 - \Omega_2^2)^2 - 4\Omega_1^2 \Omega_2^2 - \Omega_3^4}{8\Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{3e^{-\Phi}}{8\Omega_3^4} (\nabla h_1)^2 \\
& + \frac{e^\Phi}{4\Omega_2^2 \Omega_3^2} (\nabla h_2)^2 - \frac{e^{-\Phi}}{8\Omega_2^4} (\nabla h_3)^2 + \frac{(h_1 - h_3)^2 e^{-\Phi}}{16\Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{3h_2^2 e^\Phi}{8\Omega_1^2 \Omega_3^4} - \frac{(h_2 - \frac{P}{9})^2 e^\Phi}{8\Omega_1^2 \Omega_2^4} \\
& + \frac{1}{4\Omega_1^2 \Omega_2^4 \Omega_3^4} \left(4\Omega_0 + h_2 (h_3 - h_1) + \frac{1}{9} P h_1 \right)^2, \tag{2.11}
\end{aligned}$$

$$\begin{aligned}
R_{5\mu\nu} = & \Omega_1^{-1} \nabla_\mu \nabla_\nu \Omega_1 + 2\Omega_2^{-1} \nabla_\mu \nabla_\nu \Omega_2 + 2\Omega_3^{-1} \nabla_\mu \nabla_\nu \Omega_3 + \frac{1}{2} \nabla_\mu \Phi \nabla_\nu \Phi \\
& + \frac{e^{-\Phi}}{2\Omega_3^4} \nabla_\mu h_1 \nabla_\nu h_1 + \frac{e^\Phi}{\Omega_2^2 \Omega_3^2} \nabla_\mu h_2 \nabla_\nu h_2 + \frac{e^{-\Phi}}{2\Omega_2^4} \nabla_\mu h_3 \nabla_\nu h_3 - \frac{1}{8} g_{\mu\nu} \left[\right. \\
& e^{-\Phi} \left(\frac{(h_1 - h_3)^2}{2\Omega_1^2 \Omega_2^2 \Omega_3^2} + \frac{1}{\Omega_3^4} (\nabla h_1)^2 + \frac{1}{\Omega_2^4} (\nabla h_3)^2 \right) + e^\Phi \left(\frac{2}{\Omega_2^2 \Omega_3^2} (\nabla h_2)^2 \right. \\
& \left. \left. + \frac{1}{\Omega_1^2 \Omega_2^4} \left(h_2 - \frac{P}{9} \right)^2 + \frac{1}{\Omega_1^2 \Omega_3^4} h_2^2 \right) + \frac{2}{\Omega_1^2 \Omega_2^4 \Omega_3^4} \left(4\Omega_0 + h_2 (h_3 - h_1) + \frac{1}{9} P h_1 \right)^2 \right] .
\end{aligned} \tag{2.12}$$

We explicitly verified that equations (2.5)-(2.12) are equivalent to type IIB supergravity equations of motion provided the uplift is given by:

$$ds_{10}^2 = g_{\mu\nu}(y) dy^\mu dy^\nu + \Omega_1^2(y) g_5^2 + \Omega_2^2(y) [g_3^2 + g_4^2] + \Omega_3^2(y) [g_1^2 + g_2^2] , \tag{2.13}$$

for the 10-dimensional Einstein frame metric, and

$$\begin{aligned}
B_2 = & h_1(y) g_1 \wedge g_2 + h_3(y) g_3 \wedge g_4 , \\
F_3 = & \frac{1}{9} P g_5 \wedge g_3 \wedge g_4 + h_2(y) (g_1 \wedge g_2 - g_3 \wedge g_4) \wedge g_5 \\
& + (g_1 \wedge g_3 + g_2 \wedge g_4) \wedge d(h_2(y)) , \\
F_5 = & \left(1 + \star_{10} \right) \left(4\Omega_0 + h_2(y) (h_3(y) - h_1(y)) + \frac{1}{9} P h_1(y) \right) g_5 \wedge g_3 \wedge g_4 \wedge g_1 \wedge g_2 , \\
\Phi = & \Phi(y) ,
\end{aligned} \tag{2.14}$$

for the fluxes and the dilaton. In (2.13), (2.14) g_i 's are the following 1-forms on $T^{1,1}$

$$\begin{aligned}
g_1 = & \frac{\alpha^1 - \alpha^3}{\sqrt{2}} , & g_2 = & \frac{\alpha^2 - \alpha^4}{\sqrt{2}} , \\
g_3 = & \frac{\alpha^1 + \alpha^3}{\sqrt{2}} , & g_4 = & \frac{\alpha^2 + \alpha^4}{\sqrt{2}} , \\
g_5 = & \alpha^5 ,
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
\alpha^1 = & -\sin \theta_1 d\phi_1 , & \alpha^2 = & d\theta_1 , \\
\alpha^3 = & \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2 , \\
\alpha^4 = & \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2 , \\
\alpha^5 = & d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 .
\end{aligned} \tag{2.16}$$

2.1 χ SB fluctuations in $SU(2) \times SU(2) \times U(1)$ invariant states of cascading gauge theory

In what follows we will be interested in the spectrum of χ SB fluctuations about chirally-symmetric states of cascading gauge theory. These chirally-symmetric states are described by the gravitational configurations of (2.1) subject to constraints

$$h_1 = h_3, \quad h_2 = \frac{P}{18}, \quad \Omega_2 = \Omega_3. \quad (2.17)$$

Introducing

$$\begin{aligned} h_1 &= \frac{1}{P} \left(\frac{K_1}{12} - 36\Omega_0 \right), & h_2 &= \frac{P}{18} K_2, & h_3 &= \frac{1}{P} \left(\frac{K_3}{12} - 36\Omega_0 \right), \\ \Omega_1 &= \frac{1}{3} f_c^{1/2} h^{1/4}, & \Omega_2 &= \frac{1}{\sqrt{6}} f_a^{1/2} h^{1/4}, & \Omega_3 &= \frac{1}{\sqrt{6}} f_b^{1/2} h^{1/4}, \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} K_1 &= K + \delta k_1, & K_2 &= 1 + \delta k_2, & K_3 &= K - \delta k_1, \\ f_c &= f_2, & f_a &= f_3 + \delta f, & f_b &= f_3 - \delta f, \end{aligned} \quad (2.19)$$

we find the following effective action for the linearized fluctuations $\{\delta f, \delta k_1, \delta k_2\}$ [17]

$$S_{\chi\text{SB}}[\delta f, \delta k_1, \delta k_2] = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} h^{5/4} f_2^{1/2} f_3^2 \left\{ \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 \right\}, \quad (2.20)$$

$$\mathcal{L}_1 = - \frac{(\delta f)^2}{f_3^2} \left(- \frac{P^2 e^\Phi}{2f_2 h^{3/2} f_3^2} - \frac{(\nabla K)^2}{8f_3^2 h P^2 e^\Phi} - \frac{K^2}{2f_2 h^{5/2} f_3^4} \right), \quad (2.21)$$

$$\begin{aligned} \mathcal{L}_2 &= - \frac{9f_3^2 - 24f_2 f_3 + 4f_2^2}{f_2 h^{1/2} f_3^4} (\delta f)^2 + 2 \square \frac{(\delta f)^2}{f_3^2} - \left(\nabla \frac{\delta f}{f_3} \right)^2 \\ &\quad + 2 \nabla \left(\ln h^{1/4} f_3^{1/2} \right) \nabla \left(\frac{(\delta f)^2}{f_3^2} \right) + 2 \nabla \left(\ln f_2^{1/2} h^{5/4} f_3^2 \right) \nabla \left(\frac{(\delta f)^2}{f_3^2} \right), \end{aligned} \quad (2.22)$$

$$\begin{aligned} \mathcal{L}_3 &= - \frac{1}{2P^2 e^\Phi} \left(\frac{9}{2f_2 h^{3/2} f_3^2} (\delta k_1)^2 + \frac{1}{2h f_3^4} \left(3(\nabla K)^2 (\delta f)^2 + f_3^2 (\nabla \delta k_1)^2 \right. \right. \\ &\quad \left. \left. + 4f_3 \delta f \nabla K \nabla \delta k_1 \right) \right), \end{aligned} \quad (2.23)$$

$$\mathcal{L}_4 = \frac{P^2 e^\Phi}{2} \left(\frac{2}{9h f_3^2} (\nabla \delta k_2)^2 + \frac{2}{f_2 h^{3/2} f_3^4} (3(\delta f)^2 + 4f_3 \delta f \delta k_2 + f_3^3 (\delta k_2)^2) \right), \quad (2.24)$$

$$\mathcal{L}_5 = \frac{K}{f_2 h^{5/2} f_3^6} (f_3^2 \delta k_1 \delta k_2 - K (\delta f)^2). \quad (2.25)$$

3 Chirally symmetric phase of cascading gauge theory on S^3

We consider here $SU(2) \times SU(2) \times U(1) \times SO(4)$ (chirally-symmetric) states of the strongly coupled cascading gauge theory. We find it convenient to use a radial coordinate introduced in [11]:

$$ds_5^2 = g_{\mu\nu}(y)dy^\mu dy^\nu = h^{-1/2}\rho^{-2} \left(-dt^2 + f_1^2 (dS_3)^2 \right) + h^{1/2}\rho^{-2} (d\rho)^2, \quad (3.1)$$

where $(dS_3)^2$ is the metric on a round S^3 of unit size, and $h = h(\rho)$, $f_1 = f_1(\rho)$. Furthermore, we use parametrization (2.18) and denote⁸

$$f_c = f_2, \quad f_a = f_b = f_3, \quad K_1 = K_3 = K, \quad \Phi = \ln g, \quad (3.2)$$

with $f_i = f_i(\rho)$, and $K = K(\rho)$, $g = g(\rho)$.

Notice that parametrization (3.1) is not unique — the diffeomorphisms of the type

$$\begin{pmatrix} \rho \\ h \\ f_1 \\ f_2 \\ f_3 \\ K \\ g \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{\rho} \\ \hat{h} \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{K} \\ \hat{g} \end{pmatrix} = \begin{pmatrix} \rho/(1 + \alpha \rho) \\ (1 + \alpha \rho)^4 h \\ f_1 \\ (1 + \alpha \rho)^{-2} f_2 \\ (1 + \alpha \rho)^{-2} f_3 \\ K \\ g \end{pmatrix}, \quad \alpha = \text{const}, \quad (3.3)$$

preserve the general form of the metric. We can completely fix (3.3), *i.e.*, parameter α in (3.3), requiring that for a geodesically complete \mathcal{M}_5 the radial coordinate ρ extends as

$$\rho \in [0, +\infty). \quad (3.4)$$

⁸Recall that for the unbroken chiral symmetry we must set $K_2(\rho) \equiv 1$.

3.1 Equations of motion

For a background ansatz (3.1), (3.2), the equations of motion obtained from (2.5)-(2.12) take form

$$\begin{aligned}
0 = & f_1'' - \frac{(f_1')^2}{f_1} + \left(\frac{6}{\rho} - \frac{f_2'}{f_2} - \frac{3h'}{2h} - 4\frac{f_3'}{f_3} \right) f_1' + \frac{f_1(2f_3 - f_3'\rho)f_2'}{f_3\rho f_2} - \frac{3f_1(f_3')^2}{2f_3^2} + \frac{8f_1f_3'}{f_3\rho} \\
& + \frac{f_1(h')^2}{4h^2} + \frac{2f_1h'}{h\rho} + \frac{f_1(K')^2}{8ghf_3^2P^2} + \frac{f_1(g')^2}{4g^2} - \frac{f_1K^2}{4f_2f_3^4h^2\rho^2} - \frac{f_1P^2g}{2f_2f_3^2h\rho^2} + \frac{h}{f_1} \\
& + \frac{2f_1(6f_3 - f_2 - 3f_3^2)}{f_3^2\rho^2},
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
0 = & f_2'' + \frac{3f_2(f_1')^2}{f_1^2} - \left(\frac{15f_2}{f_1\rho} - \frac{9f_2'}{2f_1} - \frac{6f_3'f_2}{f_3f_1} - \frac{3h'f_2}{2hf_1} \right) f_1' - \frac{(f_2')^2}{2f_2} - \frac{3(2f_3 - f_3'\rho)f_2'}{f_3\rho} \\
& + \frac{3f_2(f_3')^2}{2f_3^2} - \frac{12f_2f_3'}{f_3\rho} - \frac{f_2(h')^2}{4h^2} - \frac{2f_2h'}{h\rho} - \frac{3f_2(K')^2}{8ghf_3^2P^2} - \frac{f_2(g')^2}{4g^2} + \frac{K^2}{4f_3^4h^2\rho^2} \\
& + \frac{3P^2g}{2f_3^2h\rho^2} - \frac{3f_2h}{f_1^2} + \frac{2f_2(7f_3^2 - 3f_2 - 6f_3)}{f_3^2\rho^2},
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
0 = & f_3'' + \frac{3f_3(f_1')^2}{f_1^2} - \left(\frac{15f_3}{f_1\rho} - \frac{3f_2'f_3}{2f_1f_2} - \frac{9f_3'}{f_1} - \frac{3h'f_3}{2hf_1} \right) f_1' - \frac{3(2f_3 - f_3'\rho)f_2'}{2\rho f_2} + \frac{5(f_3')^2}{2f_3} \\
& - \frac{15f_3'}{\rho} - \frac{f_3(h')^2}{4h^2} - \frac{2f_3h'}{h\rho} - \frac{(K')^2}{8ghf_3P^2} - \frac{f_3(g')^2}{4g^2} + \frac{K^2}{4f_2f_3^3h^2\rho^2} + \frac{P^2g}{2f_3hf_2\rho^2} - \frac{3f_3h}{f_1^2} \\
& - \frac{2(12f_3 - 3f_2 - 7f_3^2)}{f_3\rho^2},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
0 = & h'' - \frac{9h(f_1')^2}{f_1^2} + \left(\frac{39h}{\rho f_1} - \frac{3h'}{2f_1} - \frac{18f_3'h}{f_3f_1} - \frac{9f_2'h}{2f_1f_2} \right) f_1' + \left(\frac{8h}{\rho f_2} - \frac{3f_3'h}{f_3f_2} + \frac{h'}{2f_2} \right) f_2' \\
& - \frac{9h(f_3')^2}{2f_3^2} + \frac{2(16h + h'\rho)f_3'}{f_3\rho} - \frac{(h')^2}{4h} + \frac{3h'}{\rho} + \frac{5(K')^2}{8gf_3^2P^2} + \frac{3h(g')^2}{4g^2} + \frac{K^2}{4f_2f_3^4h\rho^2} \\
& - \frac{P^2g}{2f_3^2f_2\rho^2} + \frac{9h^2}{f_1^2} + \frac{2h(18f_3 - 3f_2 - 17f_3^2)}{f_3^2\rho^2},
\end{aligned} \tag{3.8}$$

$$0 = K'' + \left(\frac{3f_1'}{f_1} - \frac{g'}{g} + \frac{f_2'}{2f_2} - \frac{h'}{h} - \frac{3}{\rho} \right) K' - \frac{2gP^2K}{f_2f_3^2h\rho^2}, \tag{3.9}$$

$$0 = g'' - \frac{(g')^2}{g} + \left(\frac{3f_1'}{f_1} + \frac{f_2'}{2f_2} + \frac{2f_3'}{f_3} - \frac{3}{\rho} \right) g' + \frac{(K')^2}{4f_3^2hP^2} - \frac{g^2P^2}{f_2f_3^2h\rho^2}, \tag{3.10}$$

Additionally we have the first order constraint

$$\begin{aligned}
0 = & (K')^2 + \frac{2P^2 h f_3^2 (g')^2}{g} - \frac{24P^2 g h f_3^2 (f_1')^2}{f_1^2} + 12P^2 g f_3 \left(\frac{6f_3 h}{f_1 r} - \frac{f_2' f_3 h}{f_1 f_2} - \frac{h' f_3}{f_1} \right. \\
& \left. - \frac{4f_3' h}{f_1} \right) f_1' + \frac{8P^2 g f_3 h (2f_3 - f_3' r) f_2'}{r f_2} - 12P^2 g h (f_3')^2 + \frac{64P^2 g f_3 h f_3'}{r} + \frac{2gP^2 f_3^2 (h')^2}{h} \\
& + \frac{16gP^2 f_3^2 h'}{r} - 2gP^2 \left(\frac{K^2}{f_2 h f_3^2 r^2} - \frac{12h^2 f_3^2}{f_1^2} - \frac{48h f_3}{r^2} + \frac{8f_2 h}{r^2} + \frac{2gP^2}{f_2 r^2} + \frac{24h f_3^2}{r^2} \right).
\end{aligned} \tag{3.11}$$

We explicitly verified that the constraint (3.11) is consistent with (3.5)-(3.10).

3.2 UV asymptotics

The general UV (as $\rho \rightarrow 0$) asymptotic solution of (3.5)-(3.11) describing the symmetric phase of cascading gauge theory takes form

$$f_1 = f_0 \left(1 + \left(-\frac{K_0}{8} - \frac{1}{16} P^2 g_0 + \frac{1}{4} P^2 g_0 \ln \rho \right) \frac{\rho^2}{f_0^2} + \sum_{n=3}^{\infty} \sum_k f_{n,k} \frac{\rho^n}{f_0^n} \ln^k \rho \right), \tag{3.12}$$

$$f_2 = 1 - \alpha_{1,0} \frac{\rho}{f_0} + \left(\frac{K_0}{4} + \frac{\alpha_{1,0}^2}{4} + \frac{3}{8} P^2 g_0 - \frac{1}{2} P^2 g_0 \ln \rho \right) \frac{\rho^2}{f_0^2} + \sum_{n=3}^{\infty} \sum_k a_{n,k} \frac{\rho^n}{f_0^n} \ln^k \rho, \tag{3.13}$$

$$f_3 = 1 - \alpha_{1,0} \frac{\rho}{f_0} + \left(\frac{K_0}{4} + \frac{\alpha_{1,0}^2}{4} + \frac{5}{16} P^2 g_0 - \frac{1}{2} P^2 g_0 \ln \rho \right) \frac{\rho^2}{f_0^2} + \sum_{n=3}^{\infty} \sum_k b_{n,k} \frac{\rho^n}{f_0^n} \ln^k \rho, \tag{3.14}$$

$$\begin{aligned}
h = & \frac{1}{8} P^2 g_0 + \frac{1}{4} K_0 - \frac{1}{2} P^2 g_0 \ln \rho + \alpha_{1,0} \left(\frac{1}{2} K_0 - P^2 g_0 \ln \rho \right) \frac{\rho}{f_0} + \left(\frac{23}{288} P^4 g_0^2 - \frac{1}{8} K_0^2 \right. \\
& - \frac{1}{6} P^2 g_0 K_0 + \frac{\alpha_{1,0}^2}{8} (5K_0 - 2P^2 g_0) + \frac{1}{6} P^2 g_0 \left(3K_0 + 2P^2 g_0 - \frac{15}{2} \alpha_{1,0}^2 \right) \ln \rho \\
& \left. - \frac{1}{2} P^4 g_0^2 \ln^2 \rho \right) \frac{\rho^2}{f_0^2} + \sum_{n=3}^{\infty} \sum_k h_{n,k} \frac{\rho^n}{f_0^n} \ln^k \rho,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
K = & K_0 - 2P^2 g_0 \ln \rho - P^2 g_0 \alpha_{1,0} \frac{\rho}{f_0} + \left(\frac{1}{4} P^2 g_0 (K_0 + 3P^2 g_0 - \alpha_{1,0}^2) - \frac{1}{2} P^4 g_0^2 \ln \rho \right) \frac{\rho^2}{f_0^2} \\
& + \sum_{n=3}^{\infty} \sum_k K_{n,k} \frac{\rho^n}{f_0^n} \ln^k \rho,
\end{aligned} \tag{3.16}$$

$$g = g_0 \left(1 - \frac{1}{4} P^2 g_0 \frac{\rho^2}{f_0^2} + \sum_{n=3}^{\infty} \sum_k g_{n,k} \frac{\rho^n}{f_0^n} \ln^k \rho \right). \quad (3.17)$$

It is characterized by 9 parameters:

$$\{K_0, f_0, g_0, \alpha_{1,0}, b_{4,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\}. \quad (3.18)$$

In what follows we developed the UV expansion to order $\mathcal{O}(\rho^{12})$ inclusive.

3.3 IR asymptotics

We use a radial coordinate ρ that extends to infinity, see (3.4). Introducing

$$y \equiv \frac{1}{\rho}, \quad h^h \equiv y^{-4} h, \quad f_1^h \equiv y^{-1} f_1, \quad f_{2,3}^h \equiv y^2 f_{2,3}, \quad (3.19)$$

the general IR (as $y \rightarrow 0$) asymptotic solution of (3.5)-(3.11) describing the symmetric phase of cascading gauge theory takes form

$$f_1^h = h_0^h - \frac{g_0^h P^2 (h_0^h)^2 (f_{3,0}^h)^2 + 2(K_0^h)^2 - 8f_{2,0}^h (h_0^h)^4 (f_{3,0}^h)^2 (f_{2,0}^h - 6f_{3,0}^h)}{144(f_{3,0}^h)^4 (h_0^h)^3 f_{2,0}^h} y^2 + \sum_{n=2} f_{1,n}^h y^{2n}, \quad (3.20)$$

$$f_2^h = f_{2,0}^h - \frac{g_0^h P^2 - 8(h_0^h)^2 (f_{2,0}^h)^2}{8(f_{3,0}^h)^2 (h_0^h)^2} y^2 + \sum_{n=2} f_{2,n}^h y^{2n}, \quad (3.21)$$

$$f_3^h = f_{3,0}^h + \frac{3f_{3,0}^h - f_{2,0}^h}{2f_{3,0}^h} y^2 + \sum_{n=2} f_{3,n}^h y^{2n}, \quad (3.22)$$

$$h^h = (h_0^h)^2 - \frac{(K_0^h)^2 + g_0^h P^2 (h_0^h)^2 (f_{3,0}^h)^2}{8(h_0^h)^2 (f_{3,0}^h)^4 f_{2,0}^h} y^2 + \sum_{n=2} h_n^h y^{2n}, \quad (3.23)$$

$$K = K_0^h + \frac{g_0^h P^2 K_0^h}{4(f_{3,0}^h)^2 (h_0^h)^2 f_{2,0}^h} y^2 + \sum_{n=2} K_n^h y^{2n}, \quad (3.24)$$

$$g = g_0^h + \frac{(g_0^h)^2 P^2}{8(f_{3,0}^h)^2 (h_0^h)^2 f_{2,0}^h} y^2 + \sum_{n=2} g_n^h y^{2n}. \quad (3.25)$$

It is characterized by 5 parameters:

$$\{K_0^h, h_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\}. \quad (3.26)$$

In what follows we developed the IR expansion to order $\mathcal{O}(y^{12})$ inclusive.

3.4 Symmetries

The background geometry (3.1), (3.2) enjoys 4 distinct scaling symmetries. We now discuss these symmetries and exhibit their action on the asymptotic parameters (3.18).

■ First, we have:

$$P \rightarrow \lambda P, \quad g \rightarrow \frac{1}{\lambda} g, \quad \{\rho, f_i, h, K\} \rightarrow \{\rho, f_i, h, K\}, \quad \{y, f_i^h, h^h\} \rightarrow \{y, f_i, h^h\}, \quad (3.27)$$

which acts on the asymptotic parameters as

$$g_0 \rightarrow \frac{1}{\lambda} g_0, \quad \{K_0, f_0, \alpha_{1,0}, b_{4,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\} \rightarrow \{K_0, f_0, \alpha_{1,0}, b_{4,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\}, \quad (3.28)$$

and

$$\{K_0^h, h_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\} \rightarrow \{K_0^h, h_0^h, \lambda^{-1} g_0^h, f_{2,0}^h, f_{3,0}^h\}. \quad (3.29)$$

We can use the exact symmetry (3.27) to set

$$g_0 = 1. \quad (3.30)$$

■ Second, we have:

$$P \rightarrow \lambda P, \quad \rho \rightarrow \frac{1}{\lambda} \rho, \quad h \rightarrow \lambda^2 h, \quad K \rightarrow \lambda^2 K, \quad \{f_i, g\} \rightarrow \{f_i, g\}, \quad (3.31)$$

$$\{y, f_1^h, f_2^h, f_3^h, h^h\} \rightarrow \{\lambda y, \lambda^{-1} f_1^h, \lambda^2 f_2^h, \lambda^2 f_3^h, \lambda^{-2} h^h\},$$

which acts on the asymptotic parameters as

$$\{g_0, f_0\} \rightarrow \{g_0, f_0\}, \quad (3.32)$$

$$\alpha_{1,0} \rightarrow \lambda \alpha_{1,0}, \quad (3.33)$$

$$K_0 \rightarrow \lambda^2 \left(K_0 - 2P^2 g_0 \ln \lambda \right), \quad (3.34)$$

$$b_{4,0} \rightarrow \lambda^4 \left(b_{4,0} - \left(\frac{5}{32} P^4 g_0^2 + \frac{1}{12} P^2 g_0 K_0 \right) \ln \lambda + \frac{1}{12} P^4 g_0^2 \ln^2 \lambda \right), \quad (3.35)$$

$$g_{4,0} \rightarrow \lambda^4 \left(g_{4,0} + \left(3(a_{4,0} - b_{4,0}) - \frac{3}{64} P^2 g_0 K_0 - \frac{3}{64} P^4 g_0^2 \right) \ln \lambda \right), \quad (3.36)$$

$$a_{4,0} \rightarrow \lambda^4 \left(a_{4,0} - \left(\frac{1}{12} P^2 g_0 K_0 + \frac{3}{16} P^4 g_0^2 \right) \ln \lambda + \frac{1}{12} P^4 g_0^2 \ln^2 \lambda \right), \quad (3.37)$$

$$\begin{aligned} a_{6,0} \rightarrow \lambda^6 & \left(a_{6,0} + \left(\frac{107}{80} a_{4,0} P^2 g_0 - \frac{1}{640} P^2 g_0 K_0^2 - \frac{629}{76800} P^4 g_0^2 K_0 - \frac{101}{80} P^2 g_0 b_{4,0} \right. \right. \\ & - \frac{1}{10} P^2 g_0 g_{4,0} - \frac{1}{10} K_0 b_{4,0} + \frac{11959}{768000} P^6 g_0^3 + \frac{1}{10} a_{4,0} K_0 - \frac{97 P^2 g_0 + 120 K_0}{1920} P^2 g_0 \alpha_{1,0}^2 \Big) \ln \lambda \\ & + \left(\frac{1}{4} P^2 g_0 b_{4,0} - \frac{1}{4} a_{4,0} P^2 g_0 - \frac{623}{38400} P^6 g_0^3 + \frac{1}{1280} P^4 g_0^2 K_0 + \frac{1}{16} \alpha_{1,0}^2 P^4 g_0^2 \right) \ln^2 \lambda \\ & \left. + \frac{1}{480} P^6 g_0^3 \ln^3 \lambda \right), \end{aligned} \quad (3.38)$$

$$\begin{aligned}
a_{8,0} \rightarrow \lambda^8 & \left(a_{8,0} + \frac{1}{P^2 g_0 (70 K_0 - 141 P^2 g_0)} \left(18 K_0^2 (b_{4,0} - a_{4,0})^2 + \left(\frac{5}{16} K_0^3 (b_{4,0} - a_{4,0}) \right. \right. \right. \\
& - \frac{5706}{35} (b_{4,0} - a_{4,0})^2 K_0 + 24 K_0 g_{4,0} (b_{4,0} - a_{4,0}) + \frac{35}{2} a_{1,0}^2 K_0^2 (a_{4,0} - b_{4,0}) \Big) P^2 g_0 + (2 a_{4,0}^2 \\
& - 36 a_{4,0} (b_{4,0} - a_{4,0}) + 12 g_{4,0} (b_{4,0} - a_{4,0}) - \frac{23}{48} a_{4,0} K_0^2 + \frac{3437}{480} K_0^2 (b_{4,0} - a_{4,0}) \\
& + \frac{47193}{70} (b_{4,0} - a_{4,0})^2 - 140 a_{8,0} + \frac{9}{8} K_0^2 g_{4,0} - 8 g_{4,0}^2 + \frac{17}{2048} K_0^4 + \frac{2731}{16} a_{4,0} K_0 \alpha_{1,0}^2 \\
& + 350 \alpha_{1,0}^2 a_{6,0} - \frac{875}{4} \alpha_{1,0}^4 a_{4,0} - \frac{35}{128} \alpha_{1,0}^2 K_0^3 - \frac{35}{2} \alpha_{1,0}^2 K_0 g_{4,0} - \frac{175}{96} \alpha_{1,0}^4 K_0^2 \\
& - \frac{2521}{16} K_0 b_{4,0} \alpha_{1,0}^2 P^4 g_0^2 + \left(\frac{2051699}{100800} K_0 (b_{4,0} - a_{4,0}) - \frac{927}{560} a_{4,0} K_0 + \frac{2063}{32256} K_0^3 \right. \\
& + \frac{1049}{1680} K_0 g_{4,0} - \frac{185}{3} a_{6,0} - \frac{575}{4} a_{4,0} \alpha_{1,0}^2 - \frac{2533}{3072} \alpha_{1,0}^2 K_0^2 + \frac{33}{2} g_{4,0} \alpha_{1,0}^2 - \frac{2135}{192} \alpha_{1,0}^6 \\
& + \frac{2945}{192} \alpha_{1,0}^4 K_0 + \frac{1405}{8} b_{4,0} \alpha_{1,0}^2 \Big) P^6 g_0^3 + \left(\frac{1000999}{43200} a_{4,0} - \frac{13889}{15120} g_{4,0} + \frac{30969307}{270950400} K_0^2 \right. \\
& - \frac{3931199}{151200} b_{4,0} + \frac{8993}{288} \alpha_{1,0}^4 - \frac{995093}{645120} \alpha_{1,0}^2 K_0 \Big) P^8 g_0^4 + \left(\frac{2541334849}{4741632000} K_0 \right. \\
& - \frac{2003273}{403200} \alpha_{1,0}^2 P^{10} g_0^5 + \frac{6274690897}{14224896000} P^{12} g_0^6 \Big) \ln \lambda + \left(-\frac{36}{35} (b_{4,0} - a_{4,0})^2 \right. \\
& - \frac{3}{80} (K_0 - \frac{50}{3} \alpha_{1,0}^2) (b_{4,0} - a_{4,0}) P^2 g_0 + \left(\frac{389}{1200} a_{4,0} - \frac{1}{2016} K_0^2 - \frac{3}{140} g_{4,0} - \frac{219}{700} b_{4,0} \right. \\
& + \frac{1}{512} \alpha_{1,0}^2 K_0 + \frac{5}{192} \alpha_{1,0}^4 \Big) P^4 g_0^2 - \left(\frac{3103}{7526400} K_0 + \frac{3961}{107520} \alpha_{1,0}^2 \right) P^6 g_0^3 + \frac{19094567}{2370816000} P^8 g_0^4 \Big) \\
& \times \ln^2 \lambda + \left(\frac{1}{40} (b_{4,0} - a_{4,0}) P^4 g_0^2 + \left(\frac{47}{161280} K_0 + \frac{1}{192} \alpha_{1,0}^2 \right) P^6 g_0^3 - \frac{37889}{11289600} P^8 g_0^4 \right) \ln^3 \lambda \\
& + \frac{1}{20160} P^8 g_0^4 \ln^4 \lambda \Big),
\end{aligned} \tag{3.39}$$

and

$$\{K_0^h, h_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\} \rightarrow \{\lambda^2 K_0^h, \lambda^{-1} h_0^h, g_0^h, \lambda^2 f_{2,0}^h, \lambda^2 f_{3,0}^h\}. \tag{3.40}$$

We can use the exact symmetry (3.31) to relate different sets of $\{K_0, P\}$. For the study of perturbative in P^2/K_0 expansion we find it convenient to set $K_0 = 1$ and vary P^2 . To access the infrared properties of the theory we set $P = 1$ and vary K_0 . Notice that the two approaches connect at $\{K_0 = 1, P = 1\}$.

■ Third, we have:

$$\begin{aligned}
\rho \rightarrow \lambda \rho, \quad f_1 \rightarrow \lambda f_1, \quad \{P, f_2, f_3, h, K, g\} & \rightarrow \{P, f_2, f_3, h, K, g\}, \\
\{y, f_1^h, f_2^h, f_3^h, h^h\} & \rightarrow \{\lambda^{-1} y, \lambda^2 f_1^h, \lambda^{-2} f_2^h, \lambda^{-2} f_3^h, \lambda^4 h^h\},
\end{aligned} \tag{3.41}$$

provided we rescale the four-dimensional metric component $G_{tt} = -1 \rightarrow -\lambda^2$. This scaling symmetry acts on the asymptotic parameters as

$$\{g_0, \alpha_{1,0}, f_0\} \rightarrow \{g_0, \alpha_{1,0}, \lambda f_0\}, \quad (3.42)$$

$$K_0 \rightarrow K_0 + 2P^2 g_0 \ln \lambda, \quad (3.43)$$

$$b_{4,0} \rightarrow b_{4,0} + \left(\frac{5}{32} P^4 g_0^2 + \frac{1}{12} P^2 g_0 K_0 \right) \ln \lambda + \frac{1}{12} P^4 g_0^2 \ln^2 \lambda, \quad (3.44)$$

$$g_{4,0} \rightarrow g_{4,0} + \left(3(b_{4,0} - a_{4,0}) + \frac{3}{64} P^2 g_0 K_0 + \frac{3}{64} P^4 g_0^2 \right) \ln \lambda, \quad (3.45)$$

$$a_{4,0} \rightarrow a_{4,0} + \left(\frac{1}{12} P^2 g_0 K_0 + \frac{3}{16} P^4 g_0^2 \right) \ln \lambda + \frac{1}{12} P^4 g_0^2 \ln^2 \lambda, \quad (3.46)$$

$$\begin{aligned} a_{6,0} \rightarrow a_{6,0} &+ \left(\frac{1}{10} K_0 (b_{4,0} - a_{4,0}) + \left(-\frac{107}{80} a_{4,0} + \frac{1}{10} g_{4,0} + \frac{1}{640} K_0^2 + \frac{101}{80} b_{4,0} \right. \right. \\ &+ \left. \frac{1}{16} a_{1,0}^2 K_0 \right) P^2 g_0 + \left(\frac{629}{76800} K_0 + \frac{97}{1920} \alpha_{1,0}^2 \right) P^4 g_0^2 - \frac{11959}{768000} P^6 g_0^3 \Big) \ln \lambda \\ &+ \left(\frac{1}{4} (b_{4,0} - a_{4,0}) P^2 g_0 + \left(\frac{1}{16} \alpha_{1,0}^2 + \frac{1}{1280} K_0 \right) P^4 g_0^2 - \frac{623}{38400} P^6 g_0^3 \right) \ln^2 \lambda \\ &- \frac{1}{480} P^6 g_0^3 \ln^3 \lambda, \end{aligned} \quad (3.47)$$

$$\begin{aligned}
a_{8,0} \rightarrow a_{8,0} &+ \frac{1}{P^2 g_0 (70K_0 - 141P^2 g_0)} \left(-18K_0^2 (b_{4,0} - a_{4,0})^2 + \left(\frac{5706}{35} K_0 (b_{4,0} - a_{4,0})^2 \right. \right. \\
&- 24K_0 g_{4,0} (b_{4,0} - a_{4,0}) - \frac{5}{16} K_0^3 (b_{4,0} - a_{4,0}) - \frac{35}{2} \alpha_{1,0}^2 K_0^2 (a_{4,0} - b_{4,0}) \Big) P^2 g_0 + (8g_{4,0}^2 \\
&- 12g_{4,0} (b_{4,0} - a_{4,0}) - \frac{17}{2048} K_0^4 - 2a_{4,0}^2 + 36a_{4,0} (b_{4,0} - a_{4,0}) - \frac{47193}{70} (b_{4,0} - a_{4,0})^2 \\
&+ \frac{23}{48} K_0^2 a_{4,0} - \frac{3437}{480} K_0^2 (b_{4,0} - a_{4,0}) - \frac{9}{8} K_0^2 g_{4,0} + 140a_{8,0} - 350\alpha_{1,0}^2 a_{6,0} + \frac{2521}{16} K_0 b_{4,0} \alpha_{1,0}^2 \\
&+ \frac{875}{4} \alpha_{1,0}^4 a_{4,0} - \frac{2731}{16} a_{4,0} K_0 \alpha_{1,0}^2 + \frac{35}{128} \alpha_{1,0}^2 K_0^3 + \frac{175}{96} \alpha_{1,0}^4 K_0^2 + \frac{35}{2} \alpha_{1,0}^2 K_0 g_{4,0} \Big) P^4 g_0^2 \\
&+ \left(\frac{927}{560} a_{4,0} K_0 - \frac{2051699}{100800} K_0 (b_{4,0} - a_{4,0}) + \frac{185}{3} a_{6,0} - \frac{1049}{1680} K_0 g_{4,0} - \frac{2063}{32256} K_0^3 \right. \\
&+ \frac{2533}{3072} \alpha_{1,0}^2 K_0^2 - \frac{2945}{192} \alpha_{1,0}^4 K_0 - \frac{1405}{8} b_{4,0} \alpha_{1,0}^2 - \frac{33}{2} g_{4,0} \alpha_{1,0}^2 + \frac{2135}{192} \alpha_{1,0}^6 \\
&+ \frac{575}{4} a_{4,0} \alpha_{1,0}^2 \Big) P^6 g_0^3 + \left(\frac{13889}{15120} g_{4,0} - \frac{1000999}{43200} a_{4,0} - \frac{30969307}{270950400} K_0^2 + \frac{3931199}{151200} b_{4,0} \right. \\
&+ \frac{995093}{645120} \alpha_{1,0}^2 K_0 - \frac{8993}{288} \alpha_{1,0}^4 \Big) P^8 g_0^4 + \left(\frac{2003273}{403200} \alpha_{1,0}^2 - \frac{2541334849}{4741632000} K_0 \right) P^{10} g_0^5 \\
&- \frac{6274690897}{14224896000} P^{12} g_0^6 \Big) \ln \lambda + \left(-\frac{36}{35} (b_{4,0} - a_{4,0})^2 - \frac{3}{80} (K_0 - \frac{50}{3} \alpha_{1,0}^2) (b_{4,0} - a_{4,0}) P^2 g_0 \right. \\
&+ \left(-\frac{1}{2016} K_0^2 + \frac{389}{1200} a_{4,0} - \frac{3}{140} g_{4,0} - \frac{219}{700} b_{4,0} + \frac{5}{192} \alpha_{1,0}^4 + \frac{1}{512} \alpha_{1,0}^2 K_0 \right) P^4 g_0^2 \\
&- \left(\frac{3103}{7526400} K_0 + \frac{3961}{107520} \alpha_{1,0}^2 \right) P^6 g_0^3 + \frac{19094567}{2370816000} P^8 g_0^4 \Big) \ln^2 \lambda + \left(\frac{1}{40} (a_{4,0} - b_{4,0}) P^4 g_0^2 \right. \\
&- \left(\frac{47}{161280} K_0 + \frac{1}{192} \alpha_{1,0}^2 \right) P^6 g_0^3 + \frac{37889}{11289600} P^8 g_0^4 \Big) \ln^3 \lambda + \frac{1}{20160} P^8 g_0^4 \ln^4 \lambda, \\
\end{aligned} \tag{3.48}$$

and

$$\{K_0^h, h_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\} \rightarrow \{K_0^h, \lambda^2 h_0^h, g_0^h, \lambda^{-2} f_{2,0}^h, \lambda^{-2} f_{3,0}^h\}. \tag{3.49}$$

We can use the exact symmetry (3.41) to set

$$f_0 = 1. \tag{3.50}$$

■ Forth, we have residual diffeomorphisms (3.3) of the metric parametrization (3.1). The latter transformations act on asymptotic parameters as

$$\{g_0, f_0, K_0\} \rightarrow \{g_0, f_0, K_0\}, \tag{3.51}$$

$$\alpha_{1,0} \rightarrow \alpha_{1,0} + 2\alpha f_0, \tag{3.52}$$

$$a_{4,0} \rightarrow a_{4,0} - \frac{1}{4}P^2g_0\alpha f_0(\alpha_{1,0} + \alpha f_0), \quad (3.53)$$

$$b_{4,0} \rightarrow b_{4,0} - \frac{1}{4}P^2g_0\alpha f_0(\alpha_{1,0} + \alpha f_0), \quad (3.54)$$

$$g_{4,0} \rightarrow g_{4,0} - \frac{3}{4}P^2g_0\alpha f_0(\alpha_{1,0} + \alpha f_0), \quad (3.55)$$

$$\begin{aligned} a_{6,0} \rightarrow a_{6,0} + 3a_{4,0}\alpha f_0(\alpha f_0 + \alpha_{1,0}) - \frac{\alpha f_0}{24}(\alpha f_0 + \alpha_{1,0})(5K_0 - 3\alpha_{1,0}^2 + 3\alpha_{1,0}\alpha f_0 \\ + 3\alpha^2 f_0^2)P^2g_0 - \frac{37}{96}\alpha f_0(\alpha f_0 + \alpha_{1,0})P^4g_0^2, \end{aligned} \quad (3.56)$$

$$\begin{aligned} a_{8,0} \rightarrow a_{8,0} + \frac{\alpha f_0}{20}(\alpha f_0 + \alpha_{1,0})(9K_0a_{4,0} - 100a_{4,0}\alpha_{1,0}^2 + 100a_{4,0}\alpha_{1,0}\alpha f_0 + 100a_{4,0}\alpha^2 f_0^2 \\ - 9K_0b_{4,0} + 200a_{6,0}) - \frac{\alpha f_0}{11520}(\alpha f_0 + \alpha_{1,0})(2880\alpha_{1,0}^4 - 2880\alpha_{1,0}^3\alpha f_0 - 1920\alpha_{1,0}^2\alpha^2 f_0^2 \\ - 2920\alpha_{1,0}^2K_0 + 1920\alpha^3 f_0^3\alpha_{1,0} + 6160\alpha f_0K_0\alpha_{1,0} + 960\alpha^4 f_0^4 + 6160\alpha^2 f_0^2K_0 + 5184g_{4,0} \\ + 62568b_{4,0} - 66456a_{4,0} + 81K_0^2)P^2g_0 - \frac{\alpha f_0}{460800}(\alpha f_0 + \alpha_{1,0})(16623K_0 - 251240\alpha_{1,0}^2 \\ + 327200\alpha_{1,0}\alpha f_0 + 327200\alpha^2 f_0^2)P^4g_0^2 + \frac{82711}{1536000}\alpha f_0(\alpha f_0 + \alpha_{1,0})P^6g_0^3, \end{aligned} \quad (3.57)$$

and

$$\{K_0^h, h_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\} \rightarrow \{K_0^h, h_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\}. \quad (3.58)$$

As mentioned earlier, the diffeomorphisms (3.3) can be completely fixed requiring that

$$\lim_{\rho \rightarrow +\infty} f_1(\rho) = 0, \quad (3.59)$$

i.e., in the holographic dual to the symmetric phase of cascading gauge theory the manifold \mathcal{M}_5 geodesically completes in the interior with smooth shrinking of S^3 (see (3.1)) as $\rho \rightarrow +\infty$.

3.5 Keeping the physical parameters fixed

Cascading gauge theory on S^3 has two dimensionfull physical parameters: the strong coupling scale Λ and the scale $\mu \equiv \frac{1}{f_0}$, set by the size of the 3-sphere, and a dimensionless physical parameters P^2, g_0 . Recall that a symmetry transformation (3.41) rescales μ , and a symmetry transformation (3.31) rescales P and affects K_0 , while leaving the combination

$$\frac{K_0}{P^2g_0} - 2\ln f_0 + \ln P^2g_0 = \text{invariant} \equiv -2\ln \Lambda + 2\ln \mu = \ln \frac{\mu^2}{\Lambda^2} \quad (3.60)$$

invariant. The latter invariant defines the strong coupling scale Λ of cascading gauge theory. In particular, using the symmetry choices (3.30) and (3.50) we identify

$$\frac{K_0}{P^2} = \ln \frac{\mu^2}{\Lambda^2 P^2} \equiv \frac{1}{\delta}. \quad (3.61)$$

Notice that (3.61) is not invariant under the symmetry transformation (3.31). This is because such transformation modifies $P^2 g_0$, and thus changes the theory; (3.61) is invariant under the residual diffeomorphisms (3.3).

As defined in (3.61), a new dimensionless parameter δ is small when the IR cutoff set by the S^3 is much higher than the strong coupling scale Λ (and thus cascading gauge theory is close to be conformal). In section 3.7 we develop perturbative expansion in δ .

3.6 Numerical procedure

Although we would like to have an analytic control over the gravitational solution dual to a symmetric phase of cascading gauge theory, the relevant equations for $\{f_1, f_2, f_3, h, K, g\}$ (3.5)-(3.11) are rather complicated. Thus, we have to resort to numerical analysis. Recall that various scaling symmetries of the background equations of motion allowed us to set (see (3.30) and (3.50))

$$\lim_{\rho \rightarrow 0} g \equiv g_0 = 1, \quad \lim_{\rho \rightarrow 0} f_1 \equiv f_0 = 1. \quad (3.62)$$

While the metric parametrization (3.1) has residual diffeomorphisms (3.3), the latter are fixed once we insist on the IR asymptotics at $y \equiv \frac{1}{\rho} \rightarrow 0$ (see (3.59)). Finally, a scaling symmetry (3.31) relates different pairs $\{K_0, P\}$ so that only the ratio $\frac{K_0}{P^2} \equiv \frac{1}{\delta}$ is physically meaningful (see (3.61)). In the end, for a fixed δ , the gravitational solution is characterized by 6 parameters in the UV and 5 parameters in the IR:

$$\begin{aligned} \text{UV :} & \quad \{\alpha_{1,0}, b_{4,0}, a_{4,0}, a_{6,0}, a_{8,0}, g_{4,0}\}, \\ \text{IR :} & \quad \{K_0^h, h_0^h, g_0^h, f_{2,0}^h, f_{3,0}^h\}. \end{aligned} \quad (3.63)$$

Notice that $6+5 = 11$ is precisely the number of integration constants needed to specify a solution to (3.5)-(3.11) — we have 6 second order differential equations and a single first order differential constraint: $2 \times 6 - 1 = 11$.

In practice, we replace the second-order differential equation for f_2 (3.6) with the constraint equation (3.11), which we use to algebraically eliminate f_2' from (3.5), (3.7)-(3.10). The solution is found using the “shooting” method as detailed in [15].

Finding a “shooting” solution in 11-dimensional parameter space (3.63) is quite challenging. Thus, we start with (leading) analytic results for $\delta \ll 1$ (see section 3.7) and construct numerical solution for $(K_0 = 1, P^2)$ slowly incrementing P^2 from zero to one. Starting with the solution at $K_0 = P^2 = 1$ we slowly decrease K_0 while keeping $P^2 = 1$.

3.7 Symmetric phase of cascading gauge theory at $\frac{\mu}{\Lambda} \gg 1$

In this section we describe perturbative solution in $\delta \ll 1$ (3.61) to (3.5)-(3.11). Such gravitational backgrounds describe cascading gauge theory compactified on small S^3 , *i.e.*, the cutoff μ set by the compactification scale is well above the strong coupling scale Λ of cascading gauge theory.

In the limit $\delta \rightarrow 0$ (or equivalently $P \rightarrow 0$) the gravitational background is simply that of the Klebanov-Witten model [3]:

$$\begin{aligned} \delta = 0 : \quad f_1^{(0)} &= \frac{2}{\sqrt{4 + \hat{K}_0 \rho^2}}, \quad f_2^{(0)} = f_3^{(0)} = 1 + \frac{\hat{K}_0}{4} \rho^2, \quad h^{(0)} = \frac{4\hat{K}_0}{(4 + \hat{K}_0 \rho^2)^2}, \\ K^{(0)} &= \hat{K}_0, \quad g^{(0)} = 1, \end{aligned} \tag{3.64}$$

where \hat{K}_0 is a constant. Perturbatively, we find

$$\begin{aligned} f_i(\rho) &= f_i^{(0)} \times \sum_{j=0}^{\infty} \left(\frac{P^2}{\hat{K}_0} \right)^j f_{i,j}(\rho^2 \hat{K}_0), \quad h(\rho) = h^{(0)} \times \sum_{j=0}^{\infty} \left(\frac{P^2}{\hat{K}_0} \right)^j h_j(\rho^2 \hat{K}_0), \\ K(\rho) &= \hat{K}^{(0)} \times \sum_{j=0}^{\infty} \left(\frac{P^2}{\hat{K}_0} \right)^j K_j(\rho^2 \hat{K}_0), \quad g(\rho) = g^{(0)} \times \sum_{j=0}^{\infty} \left(\frac{P^2}{\hat{K}_0} \right)^j g_j(\rho^2 \hat{K}_0). \end{aligned} \tag{3.65}$$

Apart from technical complexity, there is no obstacle of developing perturbative solution to any order in $\frac{P^2}{\hat{K}_0}$. For our purposes it is sufficient to do so to order $\mathcal{O}\left(\frac{P^2}{\hat{K}_0}\right)$. Notice that explicit ρ dependence enters only in combination $\rho\sqrt{\hat{K}_0}$, thus, we can set $\hat{K}_0 = 1$ and reinstall explicit \hat{K}_0 dependence when necessary.

Substituting (3.65) in (3.5)-(3.11) we find to order $\mathcal{O}(\delta)$ the following equations

$$0 = f_{1,1}'' - \frac{4(\rho^2 + 3)}{\rho(4 + \rho^2)} f_{1,1}' - \frac{\rho}{2(4 + \rho^2)} f_{2,1}' - \frac{2\rho}{4 + \rho^2} f_{3,1}' - \frac{2(h_1 - 2f_{1,1})}{4 + \rho^2}, \tag{3.66}$$

$$0 = f''_{2,1} + \frac{12}{\rho} f'_{1,1} + \frac{2\rho}{4+\rho^2} f'_{2,1} + \frac{12}{\rho} f'_{3,1} + \frac{3\rho^2+16}{\rho(4+\rho^2)} h'_1 - \frac{2}{(4+\rho^2)\rho^2} (3\rho^2(2f_{1,1} - h_1) - 64f_{3,1} + 8f_{2,1} + 32K_1 - 32h_1), \quad (3.67)$$

$$0 = f''_{3,1} - \frac{\rho^2+28}{\rho(4+\rho^2)} f'_{3,1} - \frac{4}{\rho(4+\rho^2)} f'_{2,1} - \frac{24}{\rho(4+\rho^2)} f'_{1,1} + \frac{16(f_{3,1}+f_{2,1})}{(4+\rho^2)\rho^2}, \quad (3.68)$$

$$0 = h''_1 - \frac{12(\rho^2+2)}{\rho(4+\rho^2)} f'_{1,1} - \frac{3\rho^2+8}{\rho(4+\rho^2)} f'_{2,1} - \frac{4(3\rho^2+8)}{\rho(4+\rho^2)} f'_{3,1} - \frac{4(7+\rho^2)}{\rho(4+\rho^2)} h'_1 - \frac{2}{(4+\rho^2)\rho^2} (3\rho^2(h_1 - 2f_{1,1}) - 16 + 40f_{2,1} + 96h_1 + 160f_{3,1} - 96K_1), \quad (3.69)$$

$$0 = K''_1 - \frac{\rho^2+12}{\rho(4+\rho^2)} K'_1 - \frac{32}{(4+\rho^2)\rho^2}, \quad (3.70)$$

$$0 = g''_1 - \frac{3\rho^2+16}{\rho(4+\rho^2)} h'_1 - \frac{12(\rho^2+6)}{\rho(4+\rho^2)} f'_{1,1} - \frac{3\rho^2+16}{\rho(4+\rho^2)} f'_{2,1} - \frac{4(3\rho^2+16)}{\rho(4+\rho^2)} f'_{3,1} - \frac{\rho^2+12}{\rho(4+\rho^2)} g'_1 - \frac{2}{(4+\rho^2)\rho^2} (3\rho^2(h_1 - 2f_{1,1}) + 32h_1 + 32f_{3,1} + 8f_{2,1} - 32K_1), \quad (3.71)$$

along with the first order constraint

$$0 = (K'_1)^2 - \frac{12(\rho^2+6)}{\rho(4+\rho^2)} f'_{1,1} - \frac{3\rho^2+16}{\rho(4+\rho^2)} (f'_{2,1} + 4f'_{3,1} + h'_1) + \frac{2}{(4+\rho^2)\rho^2} (3\rho^2(2f_{1,1} - h_1) + 8 - 32h_1 - 32f_{3,1} + 32K_1 - 8f_{2,1}). \quad (3.72)$$

Above equations should be solved with $\mathcal{O}(\delta)$ UV and the IR boundary conditions prescribed in sections 3.2 and 3.3. Solving (3.70) we find

$$K_1 = \ln \left(\frac{1}{\rho^2} + \frac{1}{4} \right). \quad (3.73)$$

Next, we can use the constraint (3.72) to decouple the equation for g_1 :

$$0 = g''_1 - \frac{\rho^2+12}{\rho(4+\rho^2)} g'_1 - \frac{16}{(4+\rho^2)^2}. \quad (3.74)$$

We find

$$g_1 = \frac{1}{4} \ln \left(\frac{4}{\rho^2} + 1 \right) \left(\rho^2 - 2 \ln \left(\frac{4}{\rho^2} + 1 \right) \right) - \text{dilog} \left(\frac{4}{\rho^2} + 1 \right) - \frac{\pi^2}{6}. \quad (3.75)$$

The remaining equations (for $\{f_{i,1}, h_1\}$) we solved numerically. Parametrizing the asymptotics as follows

- UV, *i.e.*, $\rho \rightarrow 0$, (the independent coefficients being $\{f_{1,1,4}, f_{2,1,1}, f_{2,1,6}, f_{2,1,8}\}$):

$$f_{1,1} = \left(\frac{1}{16} + \frac{1}{4} \ln \rho\right) \rho^2 + \frac{1}{8} f_{2,1,1} \rho^3 + \left(f_{1,1,4} - \frac{1}{48} \ln \rho\right) \rho^4 - \frac{1}{32} f_{2,1,1} \rho^5 - \left(\frac{1}{4} f_{1,1,4} + \frac{31}{9216} - \frac{1}{192} \ln \rho\right) \rho^6 + \frac{1}{128} f_{2,1,1} \rho^7 + \left(\frac{1069}{921600} + \frac{1}{16} f_{1,1,4} - \frac{1}{768} \ln \rho\right) \rho^8 + \mathcal{O}(\rho^9), \quad (3.76)$$

$$f_{2,1} = f_{2,1,1} \rho + \left(\frac{3}{8} - \frac{1}{2} \ln \rho\right) \rho^2 - \frac{1}{4} f_{2,1,1} \rho^3 + \left(-\frac{5}{144} - 2f_{1,1,4} + \frac{1}{24} \ln \rho\right) \rho^4 + \frac{1}{16} f_{2,1,1} \rho^5 + \left(f_{2,1,6} - \frac{1}{80} \ln \rho\right) \rho^6 - \frac{1}{64} f_{2,1,1} \rho^7 + \left(f_{2,1,8} + \frac{11}{3360} \ln \rho\right) \rho^8 + \mathcal{O}(\rho^9), \quad (3.77)$$

$$f_{3,1} = f_{2,1,1} \rho + \left(\frac{5}{16} - \frac{1}{2} \ln \rho\right) \rho^2 - \frac{1}{4} f_{2,1,1} \rho^3 + \left(-\frac{1}{72} - 2f_{1,1,4} + \frac{1}{24} \ln \rho\right) \rho^4 + \frac{1}{16} f_{2,1,1} \rho^5 + \left(\frac{5}{8} f_{1,1,4} - \frac{1}{4} f_{2,1,6} + \frac{1123}{92160} - \frac{19}{1920} \ln \rho\right) \rho^6 - \frac{1}{64} f_{2,1,1} \rho^7 + \left(-\frac{3}{16} f_{1,1,4} + \frac{3}{8} f_{2,1,6} + f_{2,1,8} - \frac{1319}{307200} + \frac{67}{26880} \ln \rho\right) \rho^8 + \mathcal{O}(\rho^9), \quad (3.78)$$

$$h_1 = \frac{1}{2} - 2 \ln \rho - 2f_{2,1,1} \rho - \left(\frac{5}{12} - \ln \rho\right) \rho^2 + \frac{1}{2} f_{2,1,1} \rho^3 + \left(\frac{11}{1152} + 4f_{1,1,4} - \frac{1}{12} \ln \rho\right) \rho^4 - \frac{1}{8} f_{2,1,1} \rho^5 + \left(-f_{1,1,4} - \frac{113}{7200} + \frac{1}{48} \ln \rho\right) \rho^6 + \frac{1}{32} f_{2,1,1} \rho^7 + \left(\frac{11}{32} f_{1,1,4} - \frac{9}{8} f_{2,1,6} - \frac{15}{4} f_{2,1,8} + \frac{92753}{12902400} - \frac{289}{53760} \ln \rho\right) \rho^8 + \mathcal{O}(\rho^9); \quad (3.79)$$

- IR, *i.e.*, $y = \frac{1}{\rho} \rightarrow 0$, (the independent coefficients being $\{f_{1,1}^h, f_{2,1}^h, f_{3,1}^h\}$):

$$f_{i,1} = f_{i,1}^h + \mathcal{O}(y^2), \quad h_1 = 2f_{1,1}^h + \mathcal{O}(y^2), \quad (3.80)$$

we find

$$\begin{aligned} f_{1,1,4} &= -0.020200(6), \quad f_{2,1,1} = -0.606789(8), \quad f_{2,1,6} = 0.005345(5), \\ f_{2,1,8} &= -0.001621(4), \quad f_{1,1}^h = -0.274253(9), \quad f_{2,1}^h = -0.031228(4), \\ f_{3,1}^h &= -0.241360(3). \end{aligned} \quad (3.81)$$

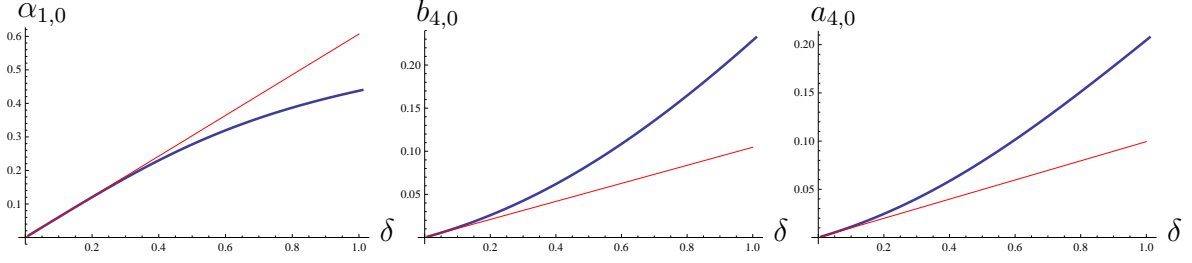


Figure 1: (Colour online) Comparison of values of UV parameters $\{\alpha_{1,0}, b_{4,0}, a_{4,0}\}$ (see (3.63)) in the range $\delta \in [0, 1]$ (blue curves) with their perturbative predictions (3.82) (red curves).

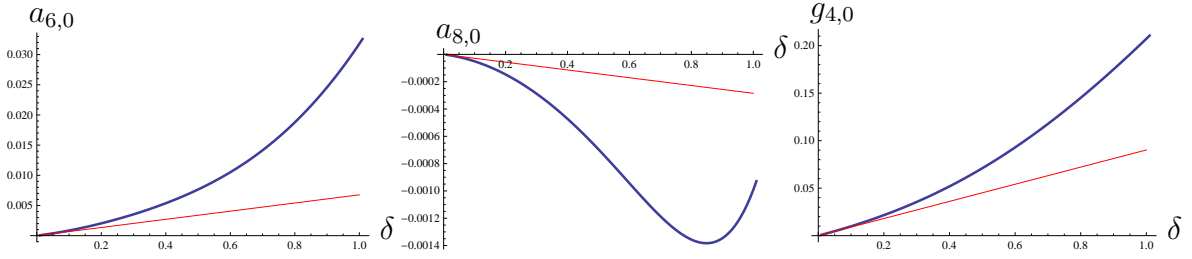


Figure 2: (Colour online) Comparison of values of UV parameters $\{a_{6,0}, a_{8,0}, g_{4,0}\}$ (see (3.63)) in the range $\delta \in [0, 1]$ (blue curves) with their perturbative predictions (3.82) (red curves).

We can now identify the leading $\mathcal{O}(\delta)$ values of general UV and IR parameters (see (3.63)):

$$\begin{aligned}
\alpha_{1,0} &= -f_{2,1,1} \delta, \quad b_{4,0} = \left(\frac{37}{576} - 2f_{1,1,4} \right) \delta, \quad a_{4,0} = \left(\frac{17}{288} - 2f_{1,1,4} \right) \delta, \\
a_{6,0} &= \left(f_{2,1,6} - \frac{1}{2}f_{1,1,4} - \frac{5}{576} \right) \delta, \quad a_{8,0} = \left(f_{2,1,8} + \frac{1}{4}f_{2,1,6} \right) \delta, \quad g_{4,0} = \left(\frac{3}{64} + \frac{1}{16} \ln 2 \right) \delta, \\
K_0^h &= 1 - 2\delta \ln 2, \quad h_0^h = 2 + 2\delta f_{1,1}^h, \quad g_0^h = 1 + \left(1 - \frac{\pi^2}{6} \right) \delta, \\
f_{2,0}^h &= \frac{1}{4} + \frac{1}{4}\delta f_{2,1}^h, \quad f_{3,0}^h = \frac{1}{4} + \frac{1}{4}\delta f_{3,1}^h,
\end{aligned} \tag{3.82}$$

where we set $K_0 = \hat{K}_0 = 1$. General relation between $\{K_0, \hat{K}_0\}$ can be obtained while acting with the symmetry (3.31):

$$K_0 = \hat{K}_0 - 2P^2 \ln \hat{K}_0 + \mathcal{O}(P^4). \tag{3.83}$$

Figures 1 and 2 compare the values of general UV parameters $\alpha_{1,0}$, $b_{4,0}$, $a_{4,0}$, $a_{6,0}$, $a_{8,0}$, $g_{4,0}$ (see (3.63)) in the range $\delta \in [0, 1]$ (blue curves) with their perturbative predictions (3.82) (red curves).

3.8 Stress-energy tensor

Holographic renormalization of cascading gauge theory was discussed in details in [11]. For a general curved boundary background \mathcal{M}_4 with

$$ds^2_{\mathcal{M}_4} = G_{ij}^{(0)} dx^i dx^j, \quad (3.84)$$

the one point correlation function of the boundary stress-energy tensor $\langle T_{ij} \rangle$ takes form

$$\begin{aligned} 8\pi G_5 \langle T_{ij} \rangle = & G_{ij}^{(0)} \left(R_{ab} R^{ab} \left(\frac{1921}{276480} \hat{p}_0^2 P^4 - \frac{1}{512} \hat{K}_0^2 + \frac{1}{96} \hat{K}_0 P^2 \hat{p}_0 \right) \right. \\ & - R^2 \left(\frac{1}{4608} \hat{K}_0^2 + \frac{337}{51840} \hat{p}_0^2 P^4 + \frac{175}{27648} \hat{K}_0 P^2 \hat{p}_0 \right) \\ & + R \left(\frac{1}{16} \hat{K}_0 \hat{a}^{(2,0)} + \frac{1}{128} P^2 \hat{p}_0 \hat{a}^{(2,0)} + \frac{5}{256} P^2 \hat{p}_0 \hat{a}^{(2,1)} \right) \\ & \left. + \square R \left(\frac{391}{82944} \hat{p}_0^2 P^4 - \frac{53}{23040} \hat{K}_0^2 + \frac{323}{46080} \hat{K}_0 P^2 \hat{p}_0 \right) \right) \\ & + R_{aijb} R^{ab} \left(\frac{17}{8640} \hat{p}_0^2 P^4 - \frac{1}{32} \hat{K}_0^2 + \frac{7}{192} \hat{K}_0 P^2 \hat{p}_0 \right) \\ & - R_i^a R_{aj} \left(\frac{1}{64} \hat{K}_0^2 + \frac{1}{256} \hat{p}_0^2 P^4 + \frac{1}{64} \hat{K}_0 P^2 \hat{p}_0 \right) \\ & + R R_{ij} \left(\frac{1691}{103680} \hat{p}_0^2 P^4 - \frac{1}{576} \hat{K}_0^2 + \frac{13}{432} \hat{K}_0 P^2 \hat{p}_0 \right) \\ & - R_{ij} \left(\frac{1}{16} P^2 \hat{p}_0 \hat{a}^{(2,1)} + \frac{1}{4} \hat{K}_0 \hat{a}^{(2,0)} \right) \\ & - \nabla_i \nabla_j R \left(\frac{2773}{207360} \hat{p}_0^2 P^4 + \frac{5}{3456} \hat{K}_0 P^2 \hat{p}_0 + \frac{7}{1152} \hat{K}_0^2 \right) \\ & + \square R_{ij} \left(-\frac{17}{17280} \hat{p}_0^2 P^4 - \frac{7}{384} \hat{K}_0 P^2 \hat{p}_0 + \frac{1}{64} \hat{K}_0^2 \right) \\ & - \nabla_i \nabla_j \hat{a}^{(2,0)} \left(\frac{1}{16} P^2 \hat{p}_0 + \frac{1}{16} \hat{K}_0 \right) + \nabla_i \nabla_j \hat{a}^{(2,1)} \left(\frac{7}{128} P^2 \hat{p}_0 + \frac{3}{64} \hat{K}_0 \right) \\ & + 2 \hat{G}_{ij}^{(4,0)} - \frac{1}{2} G_{ij}^{(0)} \hat{G}_a^{(4,0)a} + \frac{3}{2} G_{ij}^{(0)} \left(\hat{b}^{(4,0)} - \hat{a}^{(4,0)} \right) \\ & + T_{ij}^{ambiguity}, \end{aligned} \quad (3.85)$$

where

$$T_{ij}^{ambiguity} = \left(\frac{1}{2} \hat{p}_0^2 P^4 \kappa_3 + \frac{1}{2} \hat{p}_0 P^2 \kappa_2 \hat{K}_0 + \frac{1}{2} \kappa_1 \hat{K}_0^2 \right) \times \\ \left(-2 \nabla_i \nabla_j R + 6 \square R_{ij} - 12 R_{aijb} R^{ab} - 3 G_{ij}^{(0)} R_{ab} R^{ab} + R^2 G_{ij}^{(0)} - 4 R R_{ij} \right. \\ \left. - \square R G_{ij}^{(0)} \right). \quad (3.86)$$

We use $\hat{}$ to indicated asymptotic parameters used in [11]. Notice that the asymptotic expansions in [11] are done in $\hat{\alpha}_{1,0} = 0$ radial gauge. All the derivatives in (3.85) are with respect to the boundary metric (3.84); R_{aijb} , R_{ab} and R are the various Riemann tensors constructed from (3.84). $T_{ij}^{ambiguity}$, parametrized by κ_i , indicates ambiguities in renormalization prescription discussed in [11] due to defining cascading gauge theory on general manifold \mathcal{M}_4 . In a special case

$$\mathcal{M}_4 = R \times S^3, \quad (3.87)$$

the one-point correlation function of the stress-energy tensor $\langle T_{ij} \rangle$ is actually ambiguity-free⁹.

In the asymptotic UV parametrization (3.12)-(3.17) we have

$$\hat{K}_0 = K_0, \quad \hat{p}_0 = g_0, \quad G_{ij}^{(0)} dx^i dx^j = -dt^2 + f_0^2 (dS_3)^2, \\ \hat{a}^{(2,0)} = \frac{1}{f_0^2} \left(\frac{1}{4} K_0 + \frac{3}{8} P^2 g_0 \right), \quad \hat{a}^{(2,1)} = -\frac{1}{2f_0^2} P^2 g_0, \\ \hat{a}^{(4,0)} = \frac{1}{f_0^4} \left(a_{4,0} + \frac{1}{16} P^2 g_0 \alpha_{1,0}^2 \right), \quad \hat{b}^{(4,0)} = \frac{1}{f_0^4} \left(b_{4,0} + \frac{1}{16} P^2 g_0 \alpha_{1,0}^2 \right), \\ \hat{G}_{ij}^{(4,0)} dx^i dx^j = \frac{1}{f_0^2} \left(\frac{7}{576} P^4 g_0^2 + \frac{1}{16} K_0^2 - b_{4,0} - \frac{1}{16} P^2 g_0 \alpha_{1,0}^2 + \frac{55}{576} P^2 g_0 K_0 \right) (dS_3)^2. \quad (3.88)$$

Since

$$\langle T^{ij} \rangle = \mathcal{E}^s \delta_0^i \delta_0^j + \mathcal{P}^s (G^{(0)ij} + \delta_0^i \delta_0^j), \quad (3.89)$$

for the casimir energy density \mathcal{E}^s and the casimir pressure¹⁰, using (3.88) we find from

⁹There are no ambiguities in vevs of dimension-4 operators $\langle \mathcal{O}_{p_0} \rangle$ and $\langle \mathcal{O}_{K_0} \rangle$ as well — see [11] for details.

¹⁰We use the superscript s to indicate the *symmetric* phase of cascading gauge theory.

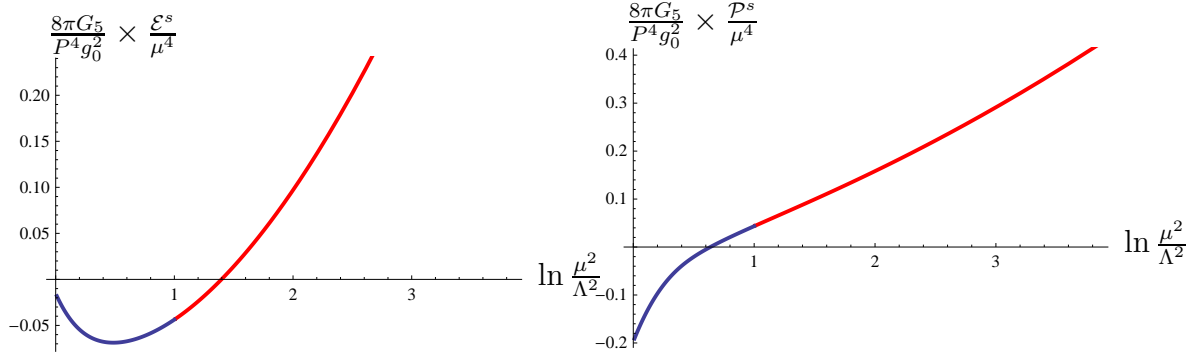


Figure 3: (Colour online) The energy density \mathcal{E}^s and the pressure \mathcal{P}^s of the chirally symmetric phase of cascading gauge theory compactified on S^3 of radius $\frac{1}{\mu}$ as a function of $\ln \frac{\mu^2}{\Lambda^2}$. The red curves are obtained from numerical solutions with $K_0 = 1$ and varying P^2 , while the blue curves are obtained with $P^2 = 1$ and varying K_0 (see section 3.6 for more details).

(3.85)

$$\begin{aligned}\mathcal{E}^s &= \frac{1}{8\pi G_5} \frac{1}{f_0^4} \left(\frac{403}{1920} P^4 g_0^2 + \frac{1}{32} K_0^2 + \frac{3}{32} K_0 P^2 g_0 - 3b_{4,0} + \frac{3}{2} a_{4,0} - \frac{3}{32} P^2 g_0 \alpha_{1,0}^2 \right), \\ \mathcal{P}^s &= \frac{1}{8\pi G_5} \frac{1}{f_0^4} \left(\frac{283}{5760} P^4 g_0^2 + \frac{1}{96} K_0^2 + \frac{1}{16} K_0 P^2 g_0 + b_{4,0} - \frac{3}{2} a_{4,0} - \frac{1}{32} P^2 g_0 \alpha_{1,0}^2 \right).\end{aligned}\tag{3.90}$$

It is instructive to understand the transformation of $\{\mathcal{E}^c, \mathcal{P}^c\}$ under the scaling symmetries (3.27), (3.31) and (3.41):

- Under (3.27), \mathcal{E}^c and \mathcal{P}^c are invariant.
- Under (3.31)

$$\{\mathcal{E}^s, \mathcal{P}^s\} \rightarrow \lambda^4 \{\mathcal{E}^s, \mathcal{P}^s\}.\tag{3.91}$$

It is easy to understand the origin of (3.91): the transformation (3.31) rescales the five-dimensional effective gravitational action (or equivalently G_5^{-1}) by λ^4 .

- Under (3.41)

$$\{\mathcal{E}^s, \mathcal{P}^s\} \rightarrow \lambda^{-4} \{\mathcal{E}^s, \mathcal{P}^s\},\tag{3.92}$$

precisely as expected, given that $f_0 \rightarrow \lambda f_0$.

- As expected, under (3.3), \mathcal{E}^s and \mathcal{P}^s are invariant.

Using (3.82), to leading order in δ (see (3.61)) we have

$$\begin{aligned}
\mathcal{E}^s &= \frac{1}{8\pi G_5} \frac{1}{f_0^4} \frac{1}{32} (K_0 - 2P^2 g_0 \ln f_0 + P^2 g_0 \ln K_0)^2 \left(1 + \left(-\frac{1}{3} + 96f_{1,1,4} \right) \delta + \mathcal{O}(\delta^2) \right) \\
&= \frac{\mu^4}{8\pi G_5} \frac{1}{32} \left(\frac{P^2 g_0}{\delta} + P^2 g_0 \ln \frac{P^2 g_0}{\delta} \right)^2 \left(1 - 2.272588(7) \delta + \mathcal{O}(\delta^2) \right), \\
\mathcal{P}^s &= \frac{1}{8\pi G_5} \frac{1}{f_0^4} \frac{1}{96} (K_0 - 2P^2 g_0 \ln f_0 + P^2 g_0 \ln K_0)^2 \left(1 + \left(\frac{11}{3} + 96f_{1,1,4} \right) \delta + \mathcal{O}(\delta^2) \right) \\
&= \frac{\mu^4}{8\pi G_5} \frac{1}{96} \left(\frac{P^2 g_0}{\delta} + P^2 g_0 \ln \frac{P^2 g_0}{\delta} \right)^2 \left(1 + 1.727411(3) \delta + \mathcal{O}(\delta^2) \right).
\end{aligned} \tag{3.93}$$

The results for the energy density \mathcal{E}^s and the pressure \mathcal{P}^s for general $\ln \frac{\mu^2}{\Lambda^2}$ are presented in figure 3. The red curves are obtained from numerical solutions with $K_0 = 1$ and varying P^2 , while the blue curves are obtained with $P^2 = 1$ and varying K_0 (see section 3.6 for more details). Notice that

$$\begin{aligned}
\frac{\mathcal{E}^s}{\mu^4} &< 0, \quad \text{once} \quad \ln \frac{\mu^2}{\Lambda^2} < 1.397064(1), \\
\frac{\mathcal{P}^s}{\mu^4} &< 0, \quad \text{once} \quad \ln \frac{\mu^2}{\Lambda^2} < 0.637321(1).
\end{aligned} \tag{3.94}$$

Likewise, we find

$$\frac{\mathcal{E}^s + 3\mathcal{P}^s}{\mu^4} < 0, \quad \text{once} \quad \ln \frac{\mu^2}{\Lambda^2} < 0.792717(5), \tag{3.95}$$

implying that cascading gauge theory compactified on sufficiently small S^3 would result in a closed inflationary Universe when coupled to four-dimensional Einstein gravity.

4 χ SB fluctuations about chirally symmetric phase of S^3 compactified cascading gauge theory

In previous sections we studied chirally symmetric phase of cascading gauge theory on S^3 . Such a phase is expected to describe the ground state of the theory once the S^3 compactification scale μ sufficiently exceeds the strong coupling scale Λ of the theory [12]. On the other hand, in the S^3 decompactification limit, *i.e.*, $\frac{\mu}{\Lambda} \rightarrow 0$, we expect the chiral symmetry to be spontaneously broken. In this section we identify χ SB fluctuations that become tachyonic once $\mu < \mu_c$, see (4.51).

4.1 Equations of motion and boundary conditions for χ SB fluctuations

Effective action for χ SB fluctuations about $SU(2) \times SU(2) \times U(1)$ symmetric states was summarized in section 2.1. Specializing to chirally symmetric states of cascading gauge theory on S^3 (3.1), and introducing

$$\delta f = \mathcal{F}(\rho) e^{-i\omega t} \Omega_L(S^3), \quad \delta k_{1,2} = \mathcal{K}_{1,2}(\rho) e^{-i\omega t} \Omega_L(S^3), \quad (4.1)$$

where $\Omega_L(S^3)$ are S^3 Laplace-Beltrami operator eigenfunctions with eigenvalues $L = \ell(\ell + 2)$ for integer ℓ

$$\Delta_{S^3} \Omega_L(S^3) = -L \Omega_L(S^3) = -\ell(\ell + 2) \Omega_L(S^3), \quad (4.2)$$

we find the following equations of motion

$$\begin{aligned} 0 = \mathcal{F}'' + \left(\frac{f_2'}{2f_2} - \frac{3}{\rho} + 2\frac{f_3'}{f_3} + 3\frac{f_1'}{f_1} \right) \mathcal{F}' - \frac{K'}{2hf_3^2gP^2} \mathcal{K}_1' + h \left(\omega^2 - \frac{L}{f_1^2} \right) \mathcal{F} - \frac{2gP^2}{h\rho^2f_2f_3^2} \mathcal{K}_2 \\ - \left(\frac{2gP^2}{h\rho^2f_2f_3^2} + \frac{(K')^2}{2hgP^2f_3^2} + \frac{9}{\rho^2f_2} - \frac{12}{\rho^2f_3} \right) \mathcal{F}, \end{aligned} \quad (4.3)$$

$$\begin{aligned} 0 = \mathcal{K}_1'' + \left(3\frac{f_1'}{f_1} - \frac{h'}{h} - \frac{3}{\rho} + \frac{f_2'}{3f_2} - \frac{g'}{g} \right) \mathcal{K}_1' + 2K' \mathcal{F}' + h \left(\omega^2 - \frac{L}{f_1^2} \right) \mathcal{K}_1 + \frac{2gP^2K}{h\rho^2f_2f_3^2} \mathcal{K}_2 \\ - \frac{9}{\rho^2f_2} \mathcal{K}_1 + \frac{4gP^2K}{h\rho^2f_2f_3^2} \mathcal{F}, \end{aligned} \quad (4.4)$$

$$\begin{aligned} 0 = \mathcal{K}_2'' + \left(3\frac{f_1'}{f_1} - \frac{h'}{h} - \frac{3}{\rho} + \frac{g'}{g} + \frac{f_2'}{2f_2} \right) \mathcal{K}_2' + h \left(\omega^2 - \frac{L}{f_1^2} \right) \mathcal{K}_2 + \frac{9K}{2h\rho^2gf_2P^2f_3^2} \mathcal{K}_1 \\ - \frac{9}{\rho^2f_2} \mathcal{K}_2 - \frac{18}{\rho^2f_2} \mathcal{F}. \end{aligned} \quad (4.5)$$

In the UV (as $\rho \rightarrow 0$) only the normalizable modes of $\{\mathcal{F}, \mathcal{K}_{1,2}\}$ can be nonzero; thus the asymptotic solution to (4.3)-(4.5) is given by¹¹

$$\mathcal{F} = \mathcal{F}_{3,0} \rho^3 + \frac{3}{2} \alpha_{1,0} \mathcal{F}_{3,0} \rho^4 + \sum_{n=5}^{\infty} \sum_k \mathcal{F}_{n,k} \rho^n \ln^k \rho, \quad (4.6)$$

$$\begin{aligned} \mathcal{K}_1 = P^2 g_0 (\mathcal{K}_{1,3,0} + 2\mathcal{F}_{3,0} \ln \rho) \rho^3 + P^2 g_0 \alpha_{1,0} \left(\frac{3}{2} \mathcal{K}_{1,3,0} + \mathcal{F}_{3,0} + 3\mathcal{F}_{3,0} \ln \rho \right) \rho^4 \\ + \sum_{n=5}^{\infty} \sum_k \mathcal{K}_{1,n,k} \rho^n \ln^k \rho, \end{aligned} \quad (4.7)$$

¹¹For the numerics we developed expansions to order $\mathcal{O}(\rho^{10})$ inclusive.

$$\begin{aligned} \mathcal{K}_2 = & \left(\frac{3}{2} \mathcal{K}_{1,3,0} - \mathcal{F}_{3,0} + 3\mathcal{F}_{3,0} \ln \rho \right) \rho^3 + \frac{9\alpha_{1,0}}{4} (\mathcal{K}_{1,3,0} + 2\mathcal{F}_{3,0} \ln \rho) \rho^4 \\ & + \sum_{n=5}^{\infty} \sum_k \mathcal{K}_{2,n,k} \rho^n \ln^k \rho. \end{aligned} \quad (4.8)$$

It is characterized by 4 parameters:

$$\{\omega^2, \mathcal{F}_{3,0}, \mathcal{F}_{7,0}, \mathcal{K}_{1,3,0}\}. \quad (4.9)$$

In the IR (as $y \equiv \frac{1}{\rho} \rightarrow 0$) the non-singular asymptotic solution to (4.3)-(4.5) is given by¹²

$$\begin{aligned} \mathcal{F} &= y^{\sqrt{1+L}-1} \times \left(\mathcal{F}_0^h + \sum_{n=1}^{\infty} \mathcal{F}_n^h y^n \right), \\ \mathcal{K}_1 &= y^{\sqrt{1+L}-1} \times \left(\mathcal{K}_{1,0}^h + \sum_{n=1}^{\infty} \mathcal{K}_{1,n}^h y^n \right), \\ \mathcal{K}_2 &= y^{\sqrt{1+L}-1} \times \left(\mathcal{K}_{2,0}^h + \sum_{n=1}^{\infty} \mathcal{K}_{2,n}^h y^n \right). \end{aligned} \quad (4.10)$$

Since equations of motion for $\{\mathcal{F}, \mathcal{K}_{1,2}\}$ are homogeneous, without loss of generality we can set

$$\mathcal{K}_{2,0}^h = 1. \quad (4.11)$$

As a result, the asymptotic expansion (4.10) is characterized by 2 additional parameters:

$$\{\mathcal{F}_0^h, \mathcal{K}_{1,0}^h\}. \quad (4.12)$$

Given (4.9) and (4.12), notice that $4 + 2 = 6$ is precisely the number of integration constants needed to specify a solution to (4.3)-(4.5) for a given chirally symmetric state of cascading gauge theory on S^3 and for a fixed L .

4.2 Spectrum of χ SB fluctuations for $\ln \frac{\mu}{\Lambda} \gg 1$

We begin exploring the spectrum of χ SB fluctuations in the regime when the S^3 compactification scale μ is much larger than the strong coupling scale Λ of cascading gauge theory. Following the perturbative expansion of the background in section 3.7, we find

¹²For the numerics we developed expansions to order $\mathcal{O}(y^{10})$ inclusive.

that perturbative in $\frac{P}{\sqrt{\hat{K}_0}}$ solution to (4.3)-(4.4) takes form

$$\begin{aligned}\mathcal{F} &= \sum_{n=1}^{\infty} \left(\frac{P}{\sqrt{\hat{K}_0}} \right)^n \mathcal{F}_{(n)}(\rho^2 \hat{K}_0), & \mathcal{K}_1 &= \hat{K}_0 \sum_{n=2}^{\infty} \left(\frac{P}{\sqrt{\hat{K}_0}} \right)^n \mathcal{K}_{1,(n)}(\rho^2 \hat{K}_0), \\ \mathcal{K}_2 &= \sum_{n=0}^{\infty} \left(\frac{P}{\sqrt{\hat{K}_0}} \right)^n \mathcal{K}_{2,(n)}(\rho^2 \hat{K}_0), & \frac{\omega^2}{\mu^2} &= \sum_{n=0}^{\infty} \left(\frac{P}{\sqrt{\hat{K}_0}} \right)^n M_{(n)}.\end{aligned}\quad (4.13)$$

4.2.1 The leading order

Introducing

$$x \equiv \rho^2 \hat{K}_0, \quad (4.14)$$

to leading order in $\frac{P}{\sqrt{\hat{K}_0}}$ we find:

$$0 = \mathcal{F}_{(1)}'' - \frac{4}{x(4+x)} \mathcal{F}_{(1)}' + \frac{12x - x^2 L - 4xL + 4xM_{(0)} + 48}{4(4+x)^2 x^2} \mathcal{F}_{(1)}, \quad (4.15)$$

$$\begin{aligned}0 &= \mathcal{K}_{1,(2)}'' - \frac{4}{x(4+x)} \mathcal{K}_{1,(2)}' + \frac{4xM_{(0)} - x^2 L - 36x - 4xL - 144}{4(4+x)^2 x^2} \mathcal{K}_{1,(2)} \\ &\quad + \frac{8}{(4+x)x^2} \mathcal{K}_{2,(0)},\end{aligned}\quad (4.16)$$

$$\begin{aligned}0 &= \mathcal{K}_{2,(0)}'' - \frac{4}{x(4+x)} \mathcal{K}_{2,(0)}' + \frac{4xM_{(0)} - x^2 L - 36x - 4xL - 144}{4(4+x)^2 x^2} \mathcal{K}_{2,(0)} \\ &\quad + \frac{18}{(4+x)x^2} \mathcal{K}_{1,(2)}.\end{aligned}\quad (4.17)$$

From now on restrict discussion to the lowest $\ell = 0$ (correspondingly $L = 0$) harmonic¹³. Solving (4.15), subject to boundary conditions (4.6) and (4.10) we find

$$\begin{aligned}\mathcal{F}_{(1)}^{[q]} &= \mathcal{A}_q \left(\frac{x}{4+x} \right)^{3/2} (4+x) \left(1 + \frac{1}{4}x \right)^{-q} {}_2F_1 \left([-q, -q+1], [-2q], 1 + \frac{1}{4}x \right), \\ M_{(0)}^{[q]} &= (2q+1)^2,\end{aligned}\quad (4.18)$$

where an integer $q = 1, 2, \dots$ labels different states in the spectrum of χ SB fluctuations, and \mathcal{A}_q is a normalization constant. From now on we restrict discussion to the lowest¹⁴ $q = 1$ state in the spectrum of χ SB fluctuations:

$$\mathcal{F}_{(1)} \equiv \mathcal{F}_{(1)}^{[1]} = \mathcal{A}_1 \left(\frac{x}{4+x} \right)^{3/2}, \quad M_{(0)} \equiv M_{(0)}^{[1]} = 9. \quad (4.19)$$

¹³Extension of the analysis to higher ℓ is straightforward.

¹⁴Extension of the analysis to higher q -states is straightforward.

Next, from (4.16) and (4.17) we find that the equation for

$$\delta\mathcal{K} \equiv \mathcal{K}_{2,(0)} - \frac{3}{2}\mathcal{K}_{1,(2)}, \quad (4.20)$$

decouples (we used the value of $M_{(0)}$ as in (4.19)):

$$0 = \delta\mathcal{K}'' - \frac{4}{x(4+x)} \delta\mathcal{K}' - \frac{12(7+x)}{(4+x)^2 x^2} \delta\mathcal{K}. \quad (4.21)$$

The only solution of (4.21) consistent with the boundary conditions (4.7), (4.8) and (4.10) is

$$\delta\mathcal{K} = 0 \quad \implies \quad \mathcal{K}_{2,(0)} = \frac{3}{2}\mathcal{K}_{1,(2)}. \quad (4.22)$$

Given (4.22), the equation for $\mathcal{K}_{1,(2)}$ (4.16), subject to the boundary conditions (4.7) and (4.10), can be solved analytically¹⁵:

$$\mathcal{K}_{1,(2)} = \frac{2}{3} \left(\frac{x}{4+x} \right)^{3/2}. \quad (4.23)$$

For completeness, we summarize the leading order results for the χ SB fluctuations

$$\mathcal{F}_{(1)} = \mathcal{A}_1 \left(\frac{x}{4+x} \right)^{3/2}, \quad \mathcal{K}_{1,(2)} = \frac{2}{3} \left(\frac{x}{4+x} \right)^{3/2}, \quad \mathcal{K}_{2,(0)} = \left(\frac{x}{4+x} \right)^{3/2}, \quad (4.24)$$

$$M_{(0)} = 9.$$

Notice that \mathcal{A}_1 is not fixed at this stage — it will be fixed at the next subleading order¹⁶.

4.2.2 The subleading order

To leading order in $\frac{P}{\sqrt{\hat{K}_0}}$, the $L = 0$ (see (4.2)) and the lowest $q = 1$ (see (4.18)) state of the linearized χ SB fluctuations is presented in (4.24). At the subleading order this state is described by the following equations

$$0 = \mathcal{F}_{(2)}'' - \frac{4}{x(x+4)} \mathcal{F}_{(2)}' + \frac{12(1+x)}{(x+4)^2 x^2} \mathcal{F}_{(2)} + \frac{32}{x(x+4)^3} \left(\frac{x}{x+4} \right)^{1/2} \\ + \frac{M_{(1)} \mathcal{A}_1 x - 32 - 8x}{(x+4)^2 x^2} \left(\frac{x}{x+4} \right)^{3/2}, \quad (4.25)$$

¹⁵The normalization constant is determined from the IR normalization of $\mathcal{K}_{2,(0)}$, see (4.11).

¹⁶We find that this pattern continues at subleading orders.

$$0 = \mathcal{K}_{1,(3)}'' - \frac{4}{x(x+4)} \mathcal{K}_{1,(3)}' - \frac{36}{(x+4)^2 x^2} \mathcal{K}_{1,(3)} + \frac{8}{(x+4)x^2} \mathcal{K}_{2,(1)} - \frac{48\mathcal{A}_1}{x(x+4)^3} \left(\frac{x}{x+4}\right)^{1/2} + \frac{2(24\mathcal{A}_1 x + M_{(1)}x + 96\mathcal{A}_1)}{3(x+4)^2 x^2} \left(\frac{x}{x+4}\right)^{3/2}, \quad (4.26)$$

$$0 = \mathcal{K}_{2,(1)}'' - \frac{4}{x(x+4)} \mathcal{K}_{2,(1)}' - \frac{36}{(x+4)^2 x^2} \mathcal{K}_{2,(1)} + \frac{18}{(x+4)x^2} \mathcal{K}_{1,(3)} + \frac{M_{(1)}x - 18\mathcal{A}_1 x - 72\mathcal{A}_1}{(x+4)^2 x^2} \left(\frac{x}{x+4}\right)^{3/2}. \quad (4.27)$$

Solving (4.25) subject to the boundary conditions (4.6) and (4.10) we find

$$\mathcal{A}_1 = \frac{8}{M_{(1)}}, \quad \mathcal{F}_{(2)} = \mathcal{A}_2 \left(\frac{x}{x+4}\right)^{3/2}, \quad (4.28)$$

where \mathcal{A}_2 is a (new) normalization constant. Introducing

$$\mathcal{K}_{1,(3)} = \left(\frac{x}{x+4}\right)^{1/2} (\mathcal{G}_1 + \mathcal{G}_2), \quad \mathcal{K}_{2,(1)} = \frac{3}{2} \left(\frac{x}{x+4}\right)^{1/2} (\mathcal{G}_1 - \mathcal{G}_2), \quad (4.29)$$

the equations of motion for $\{\mathcal{G}_1, \mathcal{G}_2\}$ decouples:

$$0 = \mathcal{G}_1'' + \frac{8}{x(x+4)^2} \mathcal{G}_1 + \frac{2(M_{(1)}^2 x + 24x - 192)}{3x(x+4)^3 M_{(1)}}, \quad (4.30)$$

$$0 = \mathcal{G}_2'' - \frac{16(6+x)}{(x+4)^2 x^2} \mathcal{G}_2 + \frac{16(7x+16)}{x(x+4)^3 M_{(1)}}. \quad (4.31)$$

Imposing the UV boundary conditions (4.7) and (4.8), as well as regularity in the IR, we find solving (4.30)

$$M_{(1)} = \mp 6\sqrt{2}, \quad \mathcal{G}_1 = \frac{x\beta_2}{4+x} \pm \frac{2\sqrt{2}x}{3(4+x)} \ln\left(1 + \frac{4}{x}\right), \quad (4.32)$$

where β_2 is (so far) an arbitrary constant; we further find solving (4.31)

$$\begin{aligned} \mathcal{G}_2 = \pm & \left(\frac{4\sqrt{2}(4+x)^2}{x^2} \left(\text{dilog}\left(1 + \frac{x}{4}\right) + \ln(4+x) \ln x - \frac{1}{2} \ln^2(4+x) + 2 \ln^2 2 \right. \right. \\ & \left. \left. - 2 \ln 2 \ln x \right) + \frac{\sqrt{2}(480x + 768 + 88x^2 + 3x^3)}{12x(4+x)} \ln\left(1 + \frac{4}{x}\right) \right. \\ & \left. + \frac{\sqrt{2}(576 + 47x^2 + 324x)}{9x(4+x)} \right). \end{aligned} \quad (4.33)$$

The \pm signs in (4.32) and (4.33) are correlated. To fix β_2 we use the normalization condition (4.11), which at the subleading order considered here implies that

$$\lim_{x \rightarrow +\infty} \mathcal{K}_{2,(1)} = 0 \quad \implies \quad \beta_2 = \mp \frac{2\sqrt{2}(3\pi^2 - 28)}{9}. \quad (4.34)$$

For completeness, we summarize the subleading order results for the χ SB fluctuations

$$\begin{aligned}
\mathcal{A}_1 &= \mp \frac{2\sqrt{2}}{3}, \quad M_{(1)} = \mp 6\sqrt{2}, \quad \mathcal{F}_{(2)} = \mathcal{A}_2 \left(\frac{x}{x+4} \right)^{3/2}, \\
\mathcal{K}_{1,(3)} &= \left(\frac{x}{x+4} \right)^{1/2} (\mathcal{G}_1 + \mathcal{G}_2), \quad \mathcal{K}_{2,(1)} = \frac{3}{2} \left(\frac{x}{x+4} \right)^{1/2} (\mathcal{G}_1 - \mathcal{G}_2), \\
\mathcal{G}_1 &= \mp \left(\frac{2\sqrt{2}(3\pi^2 - 28)}{9} \frac{x}{4+x} - \frac{2\sqrt{2}x}{3(4+x)} \ln \left(1 + \frac{4}{x} \right) \right), \\
\mathcal{G}_2 &= \pm \left(\frac{4\sqrt{2}(4+x)^2}{x^2} \left(\text{dilog} \left(1 + \frac{x}{4} \right) + \ln(4+x) \ln x - \frac{1}{2} \ln^2(4+x) + 2 \ln^2 2 \right. \right. \\
&\quad \left. \left. - 2 \ln 2 \ln x \right) + \frac{\sqrt{2}(480x + 768 + 88x^2 + 3x^3)}{12x(4+x)} \ln \left(1 + \frac{4}{x} \right) \right. \\
&\quad \left. + \frac{\sqrt{2}(576 + 47x^2 + 324x)}{9x(4+x)} \right).
\end{aligned} \tag{4.35}$$

Notice that the leading and the first subleading correction to the $L = 0$, $q = 1$ state in the spectrum of χ SB fluctuations ((4.24) and (4.35)) is determined up to a constant \mathcal{A}_2 — the latter constant will be fixed at the second subleading order.

4.2.3 The subsubleading order

To leading and the first subleading order in $\frac{P}{\sqrt{K_0}}$, the $L = 0$ (see (4.2)) and the lowest $q = 1$ (see (4.18)) state of the linearized χ SB fluctuations is presented in (4.24) and (4.35) correspondingly. At the second subleading order this state is described by the following equations

$$\begin{aligned}
0 &= \mathcal{F}_{(3)}'' - \frac{4}{x(4+x)} \mathcal{F}_{(3)}' + \frac{12(1+x)}{(4+x)^2 x^2} \mathcal{F}_{(3)} + \frac{8}{x(4+x)} \mathcal{K}_{1,(3)}' - \frac{24\sqrt{x}}{(4+x)^{5/2} M_{(1)}} h_1' \\
&+ \frac{144\sqrt{x}}{M_{(1)}(3x+16)(4+x)^{3/2}} f_{1,1}' + \frac{768 - 2 \ln 2}{(4+x)^{5/2} \sqrt{x} M_{(1)}(3x+16)} \ln \left(1 + \frac{4}{x} \right) \\
&+ \frac{144\sqrt{x}}{(4+x)^{5/2} M_{(1)}(3x+16)} f_{1,1} + \frac{24(9x+40)}{(3x+16)(4+x)^{5/2} M_{(1)} \sqrt{x}} f_{2,1} \\
&- \frac{288(x+8)}{(3x+16)(4+x)^{5/2} M_{(1)} \sqrt{x}} f_{3,1} + \frac{48(2x-64+3x^2)}{\sqrt{x} M_{(1)}(4+x)^{7/2} (3x+16)} h_1 - \frac{8}{(4+x)x^2} \mathcal{K}_{2,(1)} \\
&+ \frac{3M_{(1)}^2 \mathcal{A}_2 x^2 + 16M_{(1)}^2 \mathcal{A}_2 x - 8192 - 2368x - 192x^2 + 128xM_{(2)} + 24x^2 M_{(2)}}{\sqrt{x}(4+x)^{7/2} (3x+16) M_{(1)}},
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
0 = & \mathcal{K}_{1,(4)}'' - \frac{4}{(4+x)x} \mathcal{K}_{1,(4)}' - \frac{36}{(4+x)^2 x^2} \mathcal{K}_{1,(4)} + \frac{8}{x^2(4+x)} \mathcal{K}_{2,(2)} + \frac{M_{(1)}}{x(4+x)^2} \mathcal{K}_{1,(3)} \\
& + \frac{12\sqrt{x}}{(3x+16)(4+x)^{3/2}} f_{1,1}' - \frac{8\sqrt{x}}{(4+x)^{5/2}} f_{3,1}' - \frac{6\sqrt{x}}{(4+x)^{5/2}} h_1' - \frac{48(x+8)\ln 2}{(3x+16)\sqrt{x}(4+x)^{5/2}} \\
& - \frac{3x^3 - 768 - 288x - 8x^2}{(4+x)^{7/2}\sqrt{x}(3x+16)} \ln\left(1 + \frac{4}{x}\right) + \frac{2}{3(4+x)^{7/2}\sqrt{x}(3x+16)} \left(18f_{1,1}x^2 - 18h_1x^2 \right. \\
& + 18x^2 + 72\mathcal{A}_2x^2 - 72f_{3,1}x^2 - 9f_{2,1}x^2 + 3x^2M_{(2)} + 36g_1x^2 + 16xM_{(2)} + 120x \\
& - 324h_1x - 108f_{2,1}x + 72f_{1,1}x + 456\mathcal{A}_2x + 336g_1x - 768f_{3,1}x + 384\mathcal{A}_2 - 1152h_1 \\
& \left. - 288f_{2,1} + 768g_1 - 1920f_{3,1}\right), \tag{4.37}
\end{aligned}$$

$$\begin{aligned}
0 = & \mathcal{K}_{2,(2)}'' - \frac{4}{(4+x)x} \mathcal{K}_{2,(2)}' - \frac{36}{(4+x)^2 x^2} \mathcal{K}_{2,(2)} + \frac{18}{x^2(4+x)} \mathcal{K}_{1,(4)} + \frac{M_{(1)}}{x(4+x)^2} \mathcal{K}_{2,(1)} \\
& + \frac{18\sqrt{x}}{(3x+16)(4+x)^{3/2}} f_{1,1}' - \frac{12\sqrt{x}}{(4+x)^{5/2}} f_{3,1}' - \frac{9\sqrt{x}}{(4+x)^{5/2}} h_1' - \frac{72(x+8)\ln 2}{(3x+16)\sqrt{x}(4+x)^{5/2}} \\
& + \frac{3(768 + 288x + 40x^2 + 3x^3)}{2(4+x)^{7/2}\sqrt{x}(3x+16)} \ln\left(1 + \frac{4}{x}\right) + \frac{1}{(4+x)^{7/2}\sqrt{x}(3x+16)} \left(18f_{1,1}x^2 \right. \\
& - 1152\mathcal{A}_2 - 54\mathcal{A}_2x^2 - 324h_1x - 768f_{3,1}x - 108f_{2,1}x + 16xM_{(2)} - 336g_1x - 504\mathcal{A}_2x \\
& + 72f_{1,1}x - 18h_1x^2 - 72x - 18x^2 - 288f_{2,1} - 1152h_1 - 1920f_{3,1} - 768g_1 + 3x^2M_{(2)} \\
& \left. - 72f_{3,1}x^2 - 9f_{2,1}x^2 - 36g_1x^2\right), \tag{4.38}
\end{aligned}$$

where in order to avoid unnecessary complications of the formulas we did not substitute the explicit subleading results for $M_{(1)}$, $\mathcal{K}_{1,(3)}$ and $\mathcal{K}_{2,(1)}$, see (4.35).

We have to resort to numerics to solve (4.36)-(4.38). Notice that all the equations are coupled — at least via an undetermined so far constant \mathcal{A}_2 . To begin, notice that if $\mathcal{F}_{(3)}$ is a solution to (4.36)-(4.38) subject to the boundary conditions (4.6)-(4.8) and (4.10), (4.11), so is the combination

$$\mathcal{F}_{(3)} + \mathcal{A}_3 \left(\frac{x}{x+4} \right)^{3/2}, \tag{4.39}$$

for an arbitrary constant \mathcal{A}_3 . This constant plays the role of \mathcal{A}_1 in (4.24) and the role of \mathcal{A}_2 in (4.35) — it is fixed at the third subleading order in the perturbative expansion (4.13). Notice that the shift (4.39) adjusts the coefficients of $x^{3/2}$ in the UV expansion (4.6). Thus, up to a zero mode (4.39), a particular solution to (4.36)-(4.38) has the

following UV asymptotics

$$\mathcal{F}_{(3)} = -\frac{3f_{2,1,1}}{2M_{(1)}} x^2 + \sum_{n=5}^{\infty} \sum_{k=0}^1 \mathcal{F}_{(3),n,k} x^{n/2} \ln^k x, \quad (4.40)$$

$$\begin{aligned} \mathcal{K}_{1,(4)} = & \left(\frac{1}{12} \mathcal{A}_2 + \frac{2}{3} \mathcal{K}_{2,(2),3,0} + \frac{1}{8} \mathcal{A}_2 \ln x \right) x^{3/2} - \frac{1}{8} f_{2,1,1} x^2 \\ & + \sum_{n=5}^{\infty} \sum_{k=0}^1 \mathcal{K}_{1,(4),n,k} x^{n/2} \ln^k x, \end{aligned} \quad (4.41)$$

$$\mathcal{K}_{2,(2)} = \left(\mathcal{K}_{2,(2),3,0} + \frac{3}{16} \mathcal{A}_2 \ln x \right) x^{3/2} - \frac{3}{16} f_{2,1,1} x^2 + \sum_{n=5}^{\infty} \sum_{k=0}^1 \mathcal{K}_{2,(2),n,k} x^{n/2} \ln^k x. \quad (4.42)$$

Asymptotics (4.40)-(4.42) are completely determined by 4 parameters:

$$\{ \mathcal{A}_2, M_{(2)}, \mathcal{K}_{2,(2),3,0}, \mathcal{K}_{1,(4),7,0} \}. \quad (4.43)$$

In the IR (as $y \equiv \frac{1}{x} \rightarrow 0$) the asymptotic solution to (4.36)-(4.38) is given by

$$\begin{aligned} \mathcal{F}_{(3)} = & \mathcal{F}_{(3)}^h - \frac{1}{2M_{(1)}} \left(72f_{2,1}^h - 96f_{3,1}^h + 12\mathcal{F}_{(3)}^h M_{(1)} + 144f_{1,1}^h + M_{(1)}^2 \mathcal{A}_2 + 8M_{(2)} - 64 \right) y \\ & + \sum_{n=2}^{\infty} \mathcal{F}_{(3),n}^h y^n, \end{aligned} \quad (4.44)$$

$$\begin{aligned} \mathcal{K}_{1,(4)} = & \mathcal{K}_{1,(4)}^h + \left(8 \ln 2 - \frac{1}{3} M_2 + 8f_{3,1}^h + f_{2,1}^h - 8\mathcal{A}_2 + \frac{212}{3} - \frac{22}{3} \pi^2 + 2f_{1,1}^h \right) y \\ & + \sum_{n=2}^{\infty} \mathcal{K}_{1,(4),n}^h y^n, \end{aligned} \quad (4.45)$$

$$\begin{aligned} \mathcal{K}_{2,(2)} = & \left(12 \ln 2 - 9\mathcal{K}_{1,(4)}^h + 3f_{1,1}^h + 6 - \pi^2 + 9\mathcal{A}_2 + \frac{3}{2} f_{2,1}^h - \frac{1}{2} M_{(2)} + 12f_{3,1}^h \right) y \\ & + \sum_{n=2}^{\infty} \mathcal{K}_{2,(2),n}^h y^n. \end{aligned} \quad (4.46)$$

It is completely determined by 2 additional parameters:

$$\{ \mathcal{F}_{(3)}^h, \mathcal{K}_{1,(4)}^h \}. \quad (4.47)$$

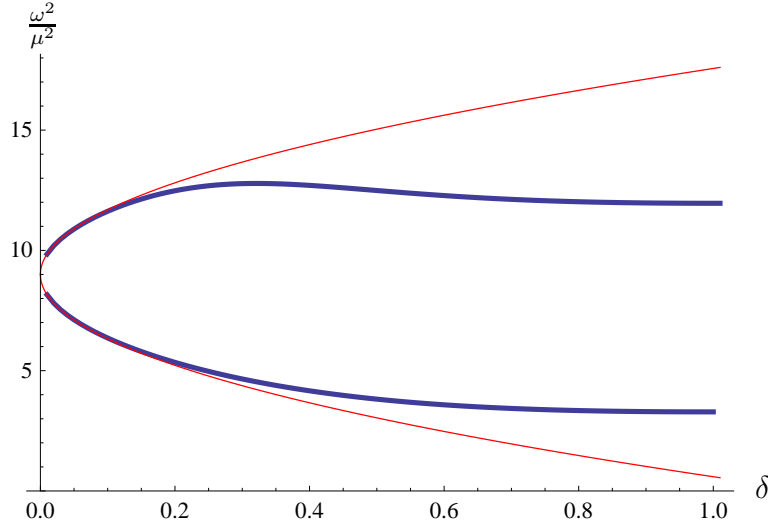


Figure 4: (Colour online) Comparison of the mass-squared ω^2 of the $L = 0$, $q = 1$ χ SB states of cascading gauge theory on S^3 as a function of $\delta = (\ln(\mu^2/(\Lambda^2 P^2 g_0)))^{-1}$ in the range $\delta \in [0, 1]$ (blue curves) with perturbative predictions (4.49) (red curves).

Numerically, we find:

$$\begin{aligned} \mathcal{A}_2 &= -0.831952(7), \quad M_{(2)} = 0.077172(8), \quad \mathcal{K}_{2,(2),3,0} = -0.000540(7), \\ \mathcal{K}_{1,(4),7,0} &= 0.000371(6), \quad \mathcal{F}_{(3)}^h = 3.116692(8), \quad \mathcal{K}_{1,(4)}^h = -0.954388(2). \end{aligned} \quad (4.48)$$

To summarize, from (4.24), (4.35) and (4.48), the mass-squared ω^2 of $L = 0$ (see (4.2)), $q = 1$ (see (4.18)) state of linearized χ SB fluctuations about chirally symmetric state of cascading gauge theory with strong coupling scale Λ , compactified on S^3 of radius $\frac{1}{\mu}$ is given by

$$\frac{\omega^2}{\mu^2} = 9 \mp 6\sqrt{2} \sqrt{\delta} + 0.077172(8) \delta + \mathcal{O}(\delta^{3/2}), \quad \delta = \left(\ln \frac{\mu^2}{\Lambda^2 P^2 g_0} \right)^{-1}. \quad (4.49)$$

4.3 Spectrum of χ SB fluctuations for general $\frac{\mu}{\Lambda}$

In this section we present results for the mass of the linearized χ SB fluctuations about the chirally symmetric state of cascading gauge theory on S^3 for general values of the S^3 compactification scale μ and the strong coupling scale Λ of cascading gauge theory. The equations of motion and the boundary conditions for the χ SB fluctuations are presented in section 4.1. A mass of a generic state in the χ SB spectrum depends on the S^3 eigenvalue L (see (4.2)) as well as on an integer $q = 1, 2, \dots$ (see (4.18)) which

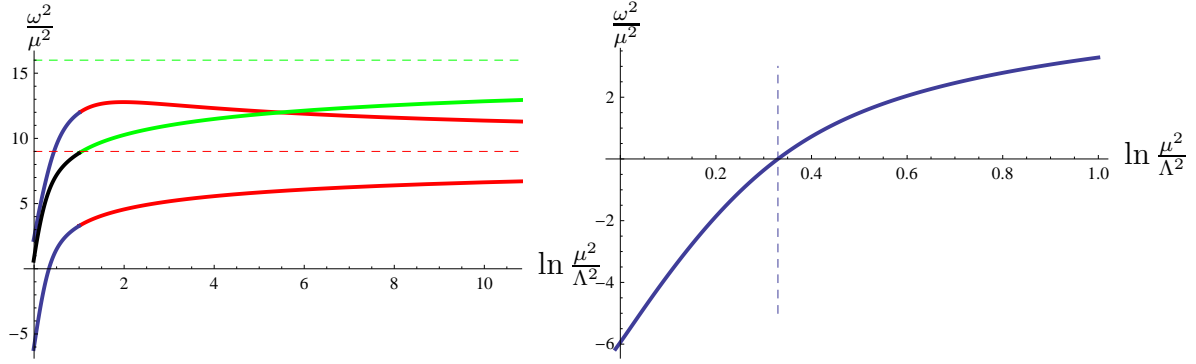


Figure 5: (Colour online) The spectrum of $L = 0$, $q = 1$ (blue and red curves) and of $L = 3$, $q = 1$ (black and green curves) states of χ SB fluctuations of cascading gauge theory on S^3 as a function of $\ln \frac{\mu^2}{\Lambda^2}$. The red and green curves are obtained from numerical solutions with $K_0 = 1$ and varying P^2 , while the blue and black curves are obtained with $P^2 = 1$ and varying K_0 (see section 3.6 for more details). The horizontal dashed lines represent the asymptotic mass-squared of the states with $L = 0$ (red curve) and $L = 3$ (green curve). The vertical blue dashed line represent the critical value μ_c (see (4.51)) below which the lighter $L = 0$, $q = 1$ state becomes tachyonic.

quantizes its radial wavefunction (4.1). For each value $\{L, q\}$ there are two branches in the spectrum arising from non-analytic dependence of a mass on $\sqrt{P^2}$. The mass of $L = 0$, $q = 1$ state was computed perturbatively in $\delta = (\ln(\mu^2/(\Lambda^2 P^2 g_0)))^{-1}$ in section 4.2, see (4.49) for the final expression.

Figure 4 compares the mass-squared $\frac{\omega^2}{\mu^2}$ of $L = 0$, $q = 1$ χ SB states in the range $\delta \in [0, 1]$ (blue curves) with perturbative predictions (4.49) (red curves).

Figure 5 presents the spectrum of $L = 0$, $q = 1$ (blue and red curves) and of $L = 3$, $q = 1$ (black and green curves) states of χ SB fluctuations of cascading gauge theory on S^3 as a function of $\ln \frac{\mu^2}{\Lambda^2}$. The red and green curves are obtained from numerical solutions with $K_0 = 1$ and varying P^2 , while the blue and black curves are obtained with $P^2 = 1$ and varying K_0 (see section 3.6 for more details). The horizontal dashed lines represent the asymptotic mass-squared of the $q = 1$ states with $L = 0$ (red curve) and $L = 3$ (green curve)

$$\lim_{\mu/\Lambda \rightarrow \infty} \left. \frac{\omega^2}{\mu^2} \right|_{L=0, q=1} = 9, \quad \lim_{\mu/\Lambda \rightarrow \infty} \left. \frac{\omega^2}{\mu^2} \right|_{L=3, q=1} = 16. \quad (4.50)$$

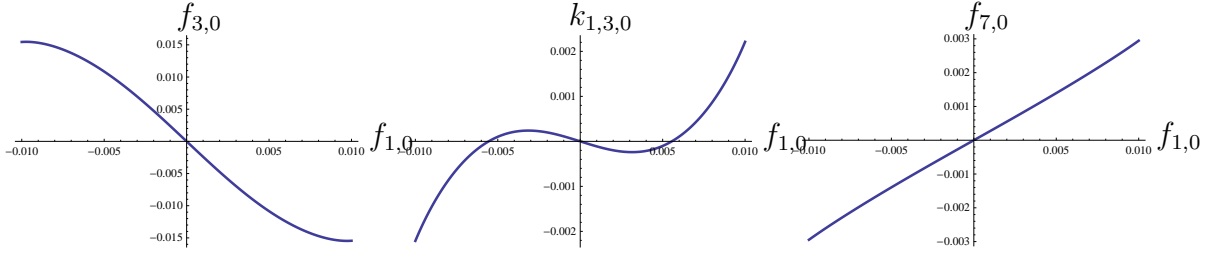


Figure 6: Expectation values of dimension-3 operators $\mathcal{O}_{3,i} \propto \{f_{3,0}, k_{1,3,0}\}$ and a dimension-7 operator $\mathcal{O}_7 \propto f_{7,0}$ of cascading gauge theory on S^3 with compactification scale $\mu_* < \mu_c$ as a function of gaugino mass deformation parameter $f_{1,0} \propto \frac{m}{\Lambda}$.

The vertical blue dashed line represent the critical value μ_c

$$\mu_c = 1.179231(5) \Lambda, \quad (4.51)$$

such that for $\mu < \mu_c$ the lighter $L = 0$, $q = 1$ χ SB state becomes tachyonic.

4.4 The end point of χ SB tachyon condensation

In section 4.3 we identified the state in the spectrum of χ SB fluctuations of cascading gauge theory on S^3 which becomes tachyonic once the S^3 compactification scale μ becomes sufficiently low, see (4.51). Notice that the mass of this state vanishes as $\mu \rightarrow \mu_c$, see figure 5. An interesting question is then whether the condensation of this tachyon signals a second-order (spontaneous) chiral symmetry breaking phase transition at $\mu = \mu_c$. We address this question following the analysis similar to the one in [17]. Notice that χ SB fluctuations about a chirally symmetric state are associated with the expectation values of two dimension-3 operators $\mathcal{O}_{3,i}$, $i = 1, 2$ (corresponding to the normalizable mode coefficients $\{\mathcal{F}_{3,0}, \mathcal{K}_{1,3,0}\}$ in (4.9)) and that of a single dimension-7 operator \mathcal{O}_7 (corresponding to the normalizable mode coefficient $\mathcal{F}_{7,0}$ in (4.9)). In a chirally symmetric state, *i.e.*, for $\mu > \mu_c$ (prior to the tachyon condensation), these vevs vanish. If the end point of the χ SB tachyon condensation is continuously (via a second-order phase transition) connected to a chirally symmetric state we expect that there is a new phase of cascading gauge theory on S^3 at $\mu < \mu_c$, such that

$$\{\mathcal{O}_{3,i} \neq 0, \mathcal{O}_7 \neq 0\} \rightarrow \{0, 0\} \quad \text{as} \quad \mu_c - \mu \rightarrow 0_+. \quad (4.52)$$

As in [17], to access this new state, we deform¹⁷ cascading gauge theory on S^3 with the compactification scale $\mu = \mu_* < \mu_c$, in practice we choose

$$\mu_* = 0.960921(1) \mu_c \quad \Longleftrightarrow \quad \{K_0 = 0.25, P^2 = 1\}, \quad (4.53)$$

by giving an explicit mass¹⁸ m to gauginous ($\mathcal{N} = 1$ fermionic superpartners of $SU(K+P) \times SU(K)$ gauge bosons). Once $m \neq 0$, such a deformation explicitly breaks chiral symmetry and generates the nonzero vevs for $\{\mathcal{O}_{3,i}, \mathcal{O}_7\}$. Without establishing the precise holographic dictionary¹⁹, it is clear that

$$\frac{m}{\Lambda} \propto f_{1,0}, \quad \mathcal{O}_{3,i} \propto \{f_{3,0}, k_{1,3,0}\}, \quad \mathcal{O}_7 \propto f_{7,0}, \quad (4.54)$$

where, at the linearized level, $f_{1,0}$ corresponds to the UV non-normalizable coefficient in \mathcal{F} (it would modify the asymptotic expansion (4.6) with a leading term $f_{1,0} \rho$), and $\{f_{3,0}, f_{7,0}, k_{1,3,0}\}$ correspond to the normalizable coefficients $\{\mathcal{F}_{3,0}, \mathcal{F}_{7,0}, \mathcal{K}_{1,3,0}\}$ in the fluctuations $\{\mathcal{F}, \mathcal{K}_1, \mathcal{K}_2\}$, see (4.6)-(4.8). The sought-after new phase of cascading gauge theory on S^3 with spontaneous breaking of chirally symmetry is obtained in the limit $\frac{m}{\Lambda} \rightarrow 0$, provided the dimension-3 and dimension-7 condensates do not vanish in this limit. We omit further details associated with discussion of the equations of motion, the appropriate boundary conditions for the holographic dual of mass-deformed cascading gauge theory on S^3 and present only the results²⁰.

Figure 6 shows expectation values of dimension-3 operators $\mathcal{O}_{3,i} \propto \{f_{3,0}, k_{1,3,0}\}$ and a dimension-7 operator $\mathcal{O}_7 \propto f_{7,0}$ of cascading gauge theory on S^3 with compactification scale $\mu_* < \mu_c$ (see (4.53)) as a function of gaugino mass deformation parameter $f_{1,0} \propto \frac{m}{\Lambda}$. Notice that all the curves are odd with respect to $f_{1,0}$ — for instance, for the range²¹ $f_{1,0} \in [-0.01, 0.01]$,

$$\left| \frac{f_{3,0}(f_{1,0})}{f_{3,0}(-f_{1,0})} + 1 \right| \sim (10^{-2} \dots 5) \times 10^{-7}, \quad (4.55)$$

and likewise for the remaining parameters. All these suggest that in the chiral limit, *i.e.*, $\frac{m}{\Lambda} \rightarrow 0$, all the χ SB condensates vanish, and the only state we find is that of (perturbatively unstable) chirally symmetric phase. Thus, condensation of the χ SB

¹⁷We consider only $SO(4)$ -invariant states.

¹⁸Generically, there are two independent mass deformations of this type, see [17].

¹⁹This can be done as in [17].

²⁰See sections 5.1.1-5.1.3 and [17] for a related detailed discussion.

²¹It is difficult numerically to reach larger values of $|f_{1,0}|$ reliably.

tachyons discussed in section 4.3 is not a signature of the second-order (spontaneous) χ SB phase transition — in other words, the end point of tachyon condensation for $\mu < \mu_c$ describes a state that can not be continuously connected to a chirally symmetric state of cascading gauge theory on S^3 .

5 Cascading gauge theory on S^3 with spontaneously broken chiral symmetry

In section 4.3 we showed that $SO(4)$ -invariant states of cascading gauge theory on S^3 with unbroken chiral symmetry are perturbatively unstable once the S^3 compactification scale $\mu < \mu_c$, see (4.51). In section (4.4) we argued that there is no $SO(4)$ -invariant phase of cascading gauge theory on S^3 with spontaneously broken chiral symmetry that is continuously connected to above chirally symmetric phase at $\mu = \mu_c$. One possibility (that we will not pursue here) is that the end point of the tachyon condensation is some $SO(4)$ non-invariant state of the theory. Another, more likely, outcome is that while the condensation end point is $SO(4)$ invariant, it is not connected via a second-order phase transition to a chirally symmetric phase. In this section we construct such a candidate state and show that it is connected via the first order phase transition to a chirally symmetric phase at $\mu = \mu_{\chi\text{SB}} > \mu_c$.

5.1 “Continuous” $R^3 \rightarrow S^3$ compactification of Klebanov-Strassler state of cascading gauge theory

$\mathcal{N} = 1$ supersymmetric ground state of cascading gauge theory on $R^{3,1}$ — referred to as Klebanov-Strassler state — spontaneously breaks chiral symmetry [2]. A natural route to construct a χ SB state of the theory on S^3 is to “compactify” Klebanov-Strassler state: $R^3 \rightarrow S^3$. We explain now how to achieve this in a “continuous” fashion.

Consider the five-dimensional metric of the type:

$$ds_5^2 = g_{\mu\nu}(y)dy^\mu dy^\nu = -c_1^2 dt^2 + c_2^2 (d\mathcal{M}_3)^2 + c_3^2 (d\rho)^2, \quad (5.1)$$

where $c_i = c_i(\rho)$, \mathcal{M}_3 is either R^3 or S^3 and $(d\mathcal{M}_3)^2$ is a standard metric on it. We will be interested in χ SB states of cascading gauge theory on \mathcal{M}_3 . One can derive equations of motion from (2.5)-(2.12). Alternatively, we can construct an effective 1-dimensional action²² from (2.1), by restricting to the metric ansatz (5.1), and the ρ -only dependence

²²Effectively, in obtaining S_1 we perform Kaluza-Klein-like reduction of S_5 on $R \times \mathcal{M}_3$.

of the scalar fields $\{\Phi, h_i, \Omega_i\}$:

$$S_5 [g_{\mu\nu}, \Omega_i, h_i, \Phi] \implies S_1 [c_i, \Omega_i, h_i, \Phi] . \quad (5.2)$$

It can be verified that equations of motion obtained from S_1 coincide with those obtained from (2.5)-(2.12), provided we vary²³ S_1 with respect to c_3 , treating it as an unconstrained field. The 1-dimensional effective action approach makes it clear that the only place where the information about \mathcal{M}_3 enters is through the evaluation of R_5 in (2.2):

$$R_5 = -\frac{6c_2''}{c_3^2 c_2} - \frac{2c_1''}{c_3^2 c_1} + \frac{2c_1' c_3'}{c_3^3 c_1} - \frac{6c_1' c_2'}{c_3^2 c_1 c_2} + \frac{6c_2' c_3'}{c_3^3 c_2} - \frac{6(c_2')^2}{c_3^2 c_2^2} + \frac{6\kappa}{c_2^2} , \quad (5.3)$$

where derivatives are with respect to ρ , and

$$\kappa = \begin{cases} 0, & \text{if } \mathcal{M}_3 = R^3 \\ 1, & \text{if } \mathcal{M}_3 = S^3 \end{cases} . \quad (5.4)$$

Even though κ takes on discrete values in (5.4), there is no obstruction in treating $\kappa \in [0, 1]$ as a continuous parameter in the effective action S_1 , thus providing a smooth, "continuous", interpolation between R^3 and S^3 . For a general value κ we denote cascading gauge theory compactification manifold $\mathcal{M}_3^{(\kappa)}$.

5.1.1 Equations of motion

As in (3.1) and (2.18) we denote

$$\begin{aligned} c_1 &= h^{-1/4} \rho^{-1}, & c_2 &= h^{-1/4} \rho^{-1} f_1, & c_3 &= h^{1/4} \rho^{-1}, & \Phi &= \ln g, \\ h_1 &= \frac{1}{P} \left(\frac{K_1}{12} - 36\Omega_0 \right), & h_2 &= \frac{P}{18} K_2, & h_3 &= \frac{1}{P} \left(\frac{K_3}{12} - 36\Omega_0 \right), \\ \Omega_1 &= \frac{1}{3} f_c^{1/2} h^{1/4}, & \Omega_2 &= \frac{1}{\sqrt{6}} f_a^{1/2} h^{1/4}, & \Omega_3 &= \frac{1}{\sqrt{6}} f_b^{1/2} h^{1/4}. \end{aligned} \quad (5.5)$$

²³This produces the first order constraint similar to (3.11).

The equations of motion obtained from $S_1 [c_i, \Omega_i, h_i, \Phi]$ are

$$\begin{aligned}
0 = & f_1'' + \frac{f_1(K_1')^2}{16hgP^2f_b^2} + \frac{f_1(K_3')^2}{16hgP^2f_a^2} + \frac{gP^2f_1(K_2')^2}{18hf_af_b} - \frac{f_1(f_a')^2}{4f_a^2} - \frac{f_1(f_b')^2}{4f_b^2} + \frac{f_1(h')^2}{4h^2} \\
& + \frac{f_1(g')^2}{4g^2} - \frac{f_c'f_1f_a'}{2f_cf_a} - \frac{f_c'f_1f_b'}{2f_cf_b} - \frac{2f_1'f_b'}{f_b} - \frac{f_1'f_c'}{f_c} - \frac{3f_1'h'}{2h} - \frac{2f_1'f_a'}{f_a} - \frac{f_1f_a'f_b'}{f_af_b} - \frac{(f_1')^2}{f_1} \\
& + \frac{2f_1f_c'}{f_c\rho} + \frac{4f_1f_a'}{f_a\rho} + \frac{2f_1h'}{h\rho} + \frac{6f_1'}{\rho} + \frac{4f_1f_b'}{f_b\rho} - \frac{2f_1f_c}{f_af_b\rho^2} - \frac{9f_1f_a}{8f_cf_b\rho^2} - \frac{9f_1f_b}{8f_cf_a\rho^2} \\
& - \frac{9f_1K_3^2}{32gf_cP^2f_af_b\rho^2} - \frac{9f_1K_1^2}{32gf_cP^2f_af_b\rho^2} + \frac{f_1gP^2K_2}{f_cf_a^2h\rho^2} - \frac{f_1gP^2K_2^2}{4f_cf_a^2h\rho^2} - \frac{f_1gP^2K_2^2}{4f_chf_b^2\rho^2} \\
& + \frac{9f_1K_1K_3}{16gf_cP^2f_af_b\rho^2} - \frac{f_1K_1^2}{4f_cf_a^2h^2f_b^2\rho^2} - \frac{f_1gP^2}{f_cf_a^2h\rho^2} - \frac{16f_1K_2^2K_1^2}{f_cf_a^2h^2f_b^2\rho^2} + \frac{f_1K_2K_1^2}{4f_cf_a^2h^2f_b^2\rho^2} \\
& - \frac{f_1K_2^2K_3^2}{16f_cf_a^2h^2f_b^2\rho^2} + \frac{f_1K_2^2K_1K_3}{8f_cf_a^2h^2f_b^2\rho^2} - \frac{f_1K_2K_3K_1}{4f_cf_a^2h^2f_b^2\rho^2} - \frac{6f_1}{\rho^2} + \frac{6f_1}{f_a\rho^2} + \frac{6f_1}{f_b\rho^2} + \frac{9f_1}{4f_c\rho^2} + \kappa \frac{h}{f_1},
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
0 = & f_c'' + \frac{3f_c(f_1')^2}{f_1^2} + \frac{f_cf_af_b'}{f_af_b} + \frac{3f_cf_1f_a'}{f_af_1} - \frac{f_c(h')^2}{4h^2} + \frac{3f_cf_1f_b'}{f_bf_1} + \frac{f_c(f_b')^2}{4f_b^2} - \frac{f_c(g')^2}{4g^2} \\
& + \frac{f_c(f_a')^2}{4f_a^2} - \frac{(f_c')^2}{2f_c} + \frac{3f_c'f_a'}{2f_a} - \frac{3f_c(K_1')^2}{16hf_b^2gP^2} - \frac{3f_c(K_3')^2}{16f_a^2hgP^2} + \frac{3f_cf_1h'}{2hf_1} + \frac{3f_c'f_b'}{2f_b} + \frac{9f_1f_c'}{2f_1} \\
& - \frac{gP^2f_c(K_2')^2}{6f_ahf_b} - \frac{6f_cf_a'}{f_a\rho} - \frac{6f_cf_b'}{f_b\rho} - \frac{15f_cf_1'}{f_1\rho} - \frac{6f_c'}{\rho} - \frac{2f_ch'}{h\rho} + \frac{K_2^2K_1^2}{16f_a^2h^2f_b^2\rho^2} - \frac{K_2^2K_1K_3}{8f_a^2h^2f_b^2\rho^2} \\
& - \frac{K_2K_1^2}{4f_a^2h^2f_b^2\rho^2} + \frac{K_2^2K_3^2}{16f_a^2h^2f_b^2\rho^2} + \frac{3gP^2}{f_a^2h\rho^2} + \frac{K_1^2}{4f_a^2h^2f_b^2\rho^2} + \frac{27K_1^2}{32f_ahf_bgP^2\rho^2} + \frac{3gP^2K_2^2}{4hf_b^2\rho^2} \\
& + \frac{3gP^2K_2^2}{4f_a^2h\rho^2} - \frac{3gP^2K_2}{f_a^2h\rho^2} + \frac{27K_3^2}{32f_ahf_bgP^2\rho^2} + \frac{K_2K_3K_1}{4f_a^2h^2f_b^2\rho^2} + \frac{45f_b}{8f_a\rho^2} - \frac{45}{4\rho^2} - \frac{6f_c}{f_a\rho^2} + \frac{45f_a}{8f_b\rho^2} \\
& - \frac{6f_c}{f_b\rho^2} - \frac{6f_c^2}{f_af_b\rho^2} + \frac{14f_c}{\rho^2} - \frac{27K_1K_3}{16f_ahf_bgP^2\rho^2} - \kappa \frac{3hf_c}{f_1^2},
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
0 = & f_a'' + \frac{f_c' f_a'}{f_c} + \frac{3f_a f_1' h'}{2h f_1} - \frac{3f_a (K_1')^2}{16h f_b^2 g P^2} + \frac{f_a (f_b')^2}{4f_b^2} + \frac{6f_1' f_a'}{f_1} + \frac{3f_a (f_1')^2}{f_1^2} + \frac{3f_a f_1' f_b'}{f_b f_1} \\
& + \frac{2f_a' f_b'}{f_b} - \frac{g P^2 (K_2')^2}{18h f_b} + \frac{(K_3')^2}{16f_a h g P^2} - \frac{f_a (g')^2}{4g^2} + \frac{(f_a')^2}{4f_a} + \frac{3f_a f_c' f_1'}{2f_1 f_c} + \frac{f_a f_c' f_b'}{2f_b f_c} - \frac{f_a (h')^2}{4h^2} \\
& - \frac{9f_a'}{\rho} - \frac{3f_a f_c'}{f_c \rho} - \frac{6f_a f_b'}{f_b \rho} - \frac{15f_a f_1'}{f_1 \rho} - \frac{2f_a h'}{h \rho} - \frac{f_a g P^2 K_2^2}{4h f_b^2 f_c \rho^2} - \frac{3g P^2 K_2}{f_a h f_c \rho^2} + \frac{3g P^2 K_2^2}{4f_a h f_c \rho^2} \\
& - \frac{K_2^2 K_1 K_3}{8f_a h^2 f_b^2 f_c \rho^2} + \frac{K_2 K_3 K_1}{4f_a h^2 f_b^2 f_c \rho^2} + \frac{9K_3^2}{32h f_b g P^2 f_c \rho^2} + \frac{27f_b}{8f_c \rho^2} - \frac{9f_a^2}{8f_b f_c \rho^2} - \frac{9f_a}{4f_c \rho^2} \\
& + \frac{9K_1^2}{32h f_b g P^2 f_c \rho^2} + \frac{K_1^2}{4f_a h^2 f_b^2 f_c \rho^2} - \frac{K_2 K_1^2}{4f_a h^2 f_b^2 f_c \rho^2} + \frac{K_2^2 K_1^2}{16f_a h^2 f_b^2 f_c \rho^2} + \frac{K_2^2 K_3^2}{16f_a h^2 f_b^2 f_c \rho^2} \\
& + \frac{3g P^2}{f_a h f_c \rho^2} - \frac{18}{\rho^2} + \frac{6f_c}{f_b \rho^2} - \frac{6f_a}{f_b \rho^2} + \frac{14f_a}{\rho^2} - \frac{9K_1 K_3}{16h f_b g P^2 f_c \rho^2} - \kappa \frac{3f_a h}{f_1^2},
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
0 = & f_b'' - \frac{3f_b (K_3')^2}{16h g f_a^2 P^2} - \frac{f_b (h')^2}{4h^2} + \frac{3f_b f_1' h'}{2h f_1} + \frac{3f_b f_c' f_1'}{2f_1 f_c} - \frac{f_b (g')^2}{4g^2} + \frac{f_c' f_b'}{f_c} + \frac{(f_b')^2}{4f_b} \\
& + \frac{(K_1')^2}{16h g f_b P^2} + \frac{f_b f_c' f_a'}{2f_a f_c} + \frac{2f_a' f_b'}{f_a} + \frac{3f_b f_1' f_a'}{f_1 f_a} + \frac{3f_b (f_1')^2}{f_1^2} + \frac{f_b (f_a')^2}{4f_a^2} - \frac{g P^2 (K_2')^2}{18h f_a} + \frac{6f_1' f_b'}{f_1} \\
& - \frac{2f_b h'}{h \rho} - \frac{15f_b f_1'}{f_1 \rho} - \frac{6f_b f_a'}{f_a \rho} - \frac{3f_b f_c'}{f_c \rho} - \frac{9f_b'}{\rho} - \frac{K_2 K_1^2}{4h^2 f_a^2 f_b f_c \rho^2} + \frac{K_2^2 K_1^2}{16h^2 f_a^2 f_b f_c \rho^2} \\
& + \frac{K_2^2 K_3^2}{16h^2 f_a^2 f_b f_c \rho^2} - \frac{g f_b P^2}{h f_a^2 f_c \rho^2} - \frac{K_2^2 K_1 K_3}{8h^2 f_a^2 f_b f_c \rho^2} + \frac{K_2 K_3 K_1}{4h^2 f_a^2 f_b f_c \rho^2} - \frac{9f_b^2}{8f_a f_c \rho^2} + \frac{27f_a}{8f_c \rho^2} - \frac{9f_b}{4f_c \rho^2} \\
& + \frac{9K_1^2}{32h g f_a P^2 f_c \rho^2} + \frac{3g P^2 K_2^2}{4h f_b f_c \rho^2} - \frac{g f_b P^2 K_2^2}{4h f_a^2 f_c \rho^2} + \frac{g f_b P^2 K_2}{h f_a^2 f_c \rho^2} + \frac{9K_3^2}{32h g f_a P^2 f_c \rho^2} + \frac{K_1^2}{4h^2 f_a^2 f_b f_c \rho^2} \\
& - \frac{6f_b}{f_a \rho^2} + \frac{6f_c}{f_a \rho^2} - \frac{18}{\rho^2} + \frac{14f_b}{\rho^2} - \frac{9K_1 K_3}{16h g f_a P^2 f_c \rho^2} - \kappa \frac{3h f_b}{f_1^2},
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
0 = & h'' - \frac{(h')^2}{4h} - \frac{9hf'_cf'_1}{2f_cf_1} - \frac{3hf'_cf'_a}{2f_cf_a} - \frac{3hf'_cf'_b}{2f_cf_b} - \frac{9hf'_1f'_b}{f_bf_1} + \frac{3h(g')^2}{4g^2} - \frac{3h(f'_b)^2}{4f_b^2} \\
& + \frac{5gP^2(K'_2)^2}{18f_af_b} - \frac{3f'_1h'}{2f_1} + \frac{f'_ah'}{f_a} + \frac{f'_bh'}{f_b} - \frac{9h(f'_1)^2}{f_1^2} - \frac{9hf'_1f'_a}{f_af_1} - \frac{3h(f'_a)^2}{4f_a^2} - \frac{3hf'_af'_b}{f_af_b} + \frac{h'f'_c}{2f_c} \\
& + \frac{5(K'_1)^2}{16f_b^2gP^2} + \frac{5(K'_3)^2}{16f_a^2gP^2} + \frac{8hf'_c}{f_c\rho} + \frac{3h'}{\rho} + \frac{16hf'_a}{f_a\rho} + \frac{16hf'_b}{f_b\rho} + \frac{39hf'_1}{f_1\rho} + \frac{K_1^2}{4f_cf_a^2hf_b^2\rho^2} \\
& + \frac{K_2^2K_1^2}{16f_cf_a^2hf_b^2\rho^2} + \frac{K_2K_3K_1}{4f_cf_a^2hf_b^2\rho^2} + \frac{gP^2K_2}{f_cf_a^2\rho^2} - \frac{K_2K_1^2}{4f_cf_a^2hf_b^2\rho^2} - \frac{gP^2K_2^2}{4f_cf_b^2\rho^2} + \frac{K_2^2K_3^2}{16f_cf_a^2hf_b^2\rho^2} \\
& - \frac{gP^2}{f_cf_a^2\rho^2} - \frac{gP^2K_2^2}{4f_cf_a^2\rho^2} - \frac{9K_3^2}{32f_cfa_fbgP^2\rho^2} - \frac{27hf_b}{8f_cfa\rho^2} - \frac{27f_ah}{8f_cfb\rho^2} + \frac{27h}{4f_c\rho^2} - \frac{9K_1^2}{32f_cfa_fbgP^2\rho^2} \\
& - \frac{K_2^2K_1K_3}{8f_cf_a^2hf_b^2\rho^2} + \frac{18h}{f_a\rho^2} - \frac{6f_ch}{f_af_b\rho^2} + \frac{18h}{f_b\rho^2} - \frac{34h}{\rho^2} + \frac{9K_1K_3}{16f_cfa_fbgP^2\rho^2} + \kappa \frac{9h^2}{f_1^2}, \tag{5.10}
\end{aligned}$$

$$\begin{aligned}
0 = & K_1'' - \frac{K'_1f'_b}{f_b} + \frac{K'_1f'_c}{2f_c} - \frac{K'_1h'}{h} + \frac{3K'_1f'_1}{f_1} - \frac{K'_1g'}{g} + \frac{K'_1f'_a}{f_a} - \frac{3K'_1}{\rho} + \frac{9f_bK_3}{2f_cfa\rho^2} \\
& + \frac{gP^2K_2^2K_3}{f_cf_a^2h\rho^2} - \frac{9f_bK_1}{2f_cfa\rho^2} - \frac{4gP^2K_1}{f_cf_a^2h\rho^2} + \frac{4gP^2K_2K_1}{f_cf_a^2h\rho^2} - \frac{gP^2K_2^2K_1}{f_cf_a^2h\rho^2} - \frac{2gP^2K_2K_3}{f_cf_a^2h\rho^2}, \tag{5.11}
\end{aligned}$$

$$\begin{aligned}
0 = & K_2'' + \frac{3K'_2f'_1}{f_1} + \frac{K'_2f'_c}{2f_c} + \frac{g'K'_2}{g} - \frac{K'_2h'}{h} - \frac{3K'_2}{\rho} - \frac{9K_3K_1}{4f_cgP^2f_ahfb\rho^2} + \frac{9K_2K_1K_3}{4f_cgP^2f_ahfb\rho^2} \\
& - \frac{9f_bK_2}{2f_cfa\rho^2} - \frac{9f_aK_2}{2f_cfb\rho^2} + \frac{9f_b}{f_cfa\rho^2} - \frac{9K_2K_1^2}{8f_cgP^2f_ahfb\rho^2} - \frac{9K_2K_3^2}{8f_cgP^2f_ahfb\rho^2} + \frac{9K_1^2}{4f_cgP^2f_ahfb\rho^2}, \tag{5.12}
\end{aligned}$$

$$\begin{aligned}
0 = & K_3'' - \frac{K'_3h'}{h} + \frac{3K'_3f'_1}{f_1} - \frac{K'_3g'}{g} + \frac{K'_3f'_c}{2f_c} + \frac{K'_3f'_b}{f_b} - \frac{K'_3f'_a}{f_a} - \frac{3K'_3}{\rho} - \frac{gP^2K_2^2K_3}{f_cf_b^2h\rho^2} \\
& - \frac{9f_aK_3}{2f_cfb\rho^2} + \frac{gP^2K_2^2K_1}{f_cf_b^2h\rho^2} + \frac{9f_aK_1}{2f_cfb\rho^2} - \frac{2gP^2K_2K_1}{f_cf_b^2h\rho^2}, \tag{5.13}
\end{aligned}$$

$$\begin{aligned}
0 = & g'' - \frac{g^2P^2(K'_2)^2}{9f_af_bh} + \frac{g'f'_b}{f_b} - \frac{(g')^2}{g} + \frac{3g'f'_1}{f_1} + \frac{(K'_1)^2}{8f_b^2hP^2} + \frac{(K'_3)^2}{8f_a^2hP^2} + \frac{g'f'_c}{2f_c} + \frac{g'f'_a}{f_a} \\
& - \frac{3g'}{\rho} - \frac{g^2P^2K_2^2}{2f_cf_b^2h\rho^2} + \frac{9K_3^2}{16f_cfa_fbh\rho^2P^2} - \frac{2g^2P^2}{f_cf_a^2h\rho^2} - \frac{9K_1K_3}{8f_cfa_fbh\rho^2P^2} + \frac{9K_1^2}{16f_cfa_fbh\rho^2P^2} \\
& - \frac{g^2P^2K_2^2}{2f_cf_a^2h\rho^2} + \frac{2g^2P^2K_2}{f_cf_a^2h\rho^2}. \tag{5.14}
\end{aligned}$$

Additionally, we have the first order constraint

$$\begin{aligned}
0 = & (K'_1)^2 f_a^2 + (K'_3)^2 f_b^2 - 4h g f_a^2 P^2 (f'_b)^2 + \frac{4}{h} g f_a^2 f_b^2 P^2 (h')^2 - \frac{24}{f_1 f_c} h g f_a^2 f_b^2 P^2 f'_c f'_1 \\
& - \frac{48}{f_1} h g f_a^2 f_b P^2 f'_1 f'_b + \frac{4}{g} h (g')^2 f_a^2 f_b^2 P^2 - 4h g f_b^2 P^2 (f'_a)^2 - 16h g f_a f_b P^2 f'_a f'_b \\
& - \frac{48}{f_1} h g f_a f_b^2 P^2 f'_1 f'_a - \frac{48}{f_1^2} h g f_a^2 f_b^2 P^2 (f'_1)^2 + \frac{8}{9} g^2 P^4 (K'_2)^2 f_a f_b - \frac{8}{f_c} h g f_a f_b^2 P^2 f'_c f'_a \\
& - \frac{8}{f_c} h g f_a^2 f_b P^2 f'_c f'_b - \frac{24}{f_1} g f_a^2 f_b^2 P^2 f'_1 h' + \frac{32}{f_c \rho} h g f_a^2 f_b^2 P^2 f'_c + \frac{64}{\rho} h g f_a^2 f_b P^2 f'_b \\
& + \frac{144}{f_1 \rho} h g f_a^2 f_b^2 P^2 f'_1 + \frac{32}{\rho} g f_a^2 f_b^2 P^2 h' + \frac{64}{\rho} h g f_a f_b^2 P^2 f'_a + \frac{16}{f_c \rho^2} g^2 P^4 f_b^2 K_2 \\
& + \frac{9}{f_c \rho^2} f_a f_b K_1 K_3 - \frac{4}{f_c \rho^2} g^2 P^4 f_b^2 K_2^2 - \frac{18}{f_c \rho^2} h g f_a^3 f_b P^2 + \frac{36}{f_c \rho^2} h g f_a^2 f_b^2 P^2 + \frac{96}{\rho^2} h g f_a f_b^2 P^2 \\
& + \frac{96}{\rho^2} h g f_a^2 f_b P^2 - \frac{96}{\rho^2} h g f_a^2 f_b^2 P^2 - \frac{4g P^2 K_1^2}{h f_c \rho^2} - \frac{4}{f_c \rho^2} g^2 P^4 K_2^2 f_a^2 - \frac{18}{f_c \rho^2} h g f_a f_b^3 P^2 \\
& + \frac{2g P^2 K_2^2 K_1 K_3}{h f_c \rho^2} - \frac{32}{\rho^2} h g f_c f_a f_b P^2 - \frac{g P^2 K_2^2 K_1^2}{h f_c \rho^2} + \frac{4g P^2 K_2 K_1^2}{h f_c \rho^2} - \frac{4g P^2 K_2 K_3 K_1}{h f_c \rho^2} \\
& - \frac{9}{2f_c \rho^2} f_a f_b K_1^2 - \frac{9}{2f_c \rho^2} f_a f_b K_3^2 - \frac{16}{f_c \rho^2} g^2 P^4 f_b^2 - \frac{g P^2 K_2^2 K_3^2}{h f_c \rho^2} + \kappa \frac{48}{f_1^2} h^2 g f_a^2 f_b^2 P^2.
\end{aligned} \tag{5.15}$$

We explicitly verified that for any value κ the constraint (5.15) is consistent with (5.6)-(5.14). Moreover, with

$$\kappa = 1, \quad f_c = f_2, \quad f_a = f_b = f_3, \quad K_1 = K_3 = K, \quad K_2 = 1, \tag{5.16}$$

equations (5.6)-(5.15) are equivalent to (3.5)-(3.11).

5.1.2 UV asymptotics

The general UV (as $\rho \rightarrow 0$) asymptotic solution of (5.6)-(5.15) describing the phase of cascading gauge theory with spontaneously broken chiral symmetry takes form

$$\begin{aligned}
f_1 = & f_0 \left(1 + \left(-\frac{\kappa}{8} K_0 - \frac{\kappa}{16} P^2 g_0 + \frac{\kappa}{4} P^2 g_0 \ln \rho \right) \frac{\rho^2}{f_0^2} + \left(\frac{\kappa}{16} \alpha_{1,0} P^2 g_0 - \frac{\kappa}{8} \alpha_{1,0} K_0 \right. \right. \\
& \left. \left. + \frac{\kappa}{4} \alpha_{1,0} P^2 g_0 \ln \rho \right) \frac{\rho^3}{f_0^3} + \sum_{n=4}^{\infty} \sum_k f_{1,n,k} \frac{\rho^n}{f_0^n} \ln^k \rho \right),
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
f_c = & 1 - \alpha_{1,0} \frac{\rho}{f_0} + \left(\frac{\kappa}{4} K_0 + \frac{\alpha_{1,0}^2}{4} + \frac{3\kappa}{8} P^2 g_0 - \frac{\kappa}{2} P^2 g_0 \ln \rho \right) \frac{\rho^2}{f_0^2} - \frac{\kappa}{4} \alpha_{1,0} P^2 g_0 \frac{\rho^3}{f_0^3} \\
& + \sum_{n=4}^{\infty} \sum_k f_{c,n,k} \frac{\rho^n}{f_0^n} \ln^k \rho,
\end{aligned} \tag{5.18}$$

$$f_a = 1 - \alpha_{1,0} \frac{\rho}{f_0} + \left(\frac{\kappa}{4} K_0 + \frac{\alpha_{1,0}^2}{4} + \frac{5\kappa}{16} P^2 g_0 - \frac{\kappa}{2} P^2 g_0 \ln \rho \right) \frac{\rho^2}{f_0^2} + \left(-\frac{\kappa}{4} \alpha_{1,0} P^2 g_0 + f_{a,3,0} \right) \frac{\rho^3}{f_0^3} + \sum_{n=4}^{\infty} \sum_k f_{a,n,k} \frac{\rho^n}{f_0^n} \ln^k \rho, \quad (5.19)$$

$$f_b = 1 - \alpha_{1,0} \frac{\rho}{f_0} + \left(\frac{\kappa}{4} K_0 + \frac{\alpha_{1,0}^2}{4} + \frac{5\kappa}{16} P^2 g_0 - \frac{\kappa}{2} P^2 g_0 \ln \rho \right) \frac{\rho^2}{f_0^2} + \left(-\frac{\kappa}{4} \alpha_{1,0} P^2 g_0 - f_{a,3,0} \right) \frac{\rho^3}{f_0^3} + \sum_{n=4}^{\infty} \sum_k f_{b,n,k} \frac{\rho^n}{f_0^n} \ln^k \rho, \quad (5.20)$$

$$h = \frac{1}{8} P^2 g_0 + \frac{1}{4} K_0 - \frac{1}{2} P^2 g_0 \ln \rho + \alpha_{1,0} \left(\frac{1}{2} K_0 - P^2 g_0 \ln \rho \right) \frac{\rho}{f_0} + \left(\frac{23\kappa}{288} P^4 g_0^2 - \frac{\kappa}{8} K_0^2 - \frac{\kappa}{6} P^2 g_0 K_0 + \frac{\alpha_{1,0}^2}{8} (5K_0 - 2P^2 g_0) + \frac{1}{6} P^2 g_0 \left(3\kappa K_0 + 2\kappa P^2 g_0 - \frac{15}{2} \alpha_{1,0}^2 \right) \ln \rho - \frac{\kappa}{2} P^4 g_0^2 \ln^2 \rho \right) \frac{\rho^2}{f_0^2} + \left(\frac{13\kappa}{32} \alpha_{1,0} P^4 g_0^2 - \frac{11}{24} P^2 g_0 \alpha_{1,0}^3 - \frac{\kappa}{4} \alpha_{1,0} K_0 P^2 g_0 + \frac{5}{8} K_0 \alpha_{1,0}^3 - \frac{3\kappa}{8} \alpha_{1,0} K_0^2 + \left(\frac{\kappa}{2} \alpha_{1,0} P^4 g_0^2 - \frac{5}{4} P^2 g_0 \alpha_{1,0}^3 + \frac{3\kappa}{2} \alpha_{1,0} K_0 P^2 g_0 \right) \ln \rho - \frac{3\kappa}{2} \alpha_{1,0} P^4 g_0^2 \ln^2 \rho \right) \frac{\rho^3}{f_0^3} + \sum_{n=4}^{\infty} \sum_k h_{n,k} \frac{\rho^n}{f_0^n} \ln^k \rho, \quad (5.21)$$

$$K_1 = K_0 - 2P^2 g_0 \ln \rho - P^2 g_0 \alpha_{1,0} \frac{\rho}{f_0} + \left(\frac{1}{4} P^2 g_0 (\kappa K_0 + 3P^2 g_0 \kappa - \alpha_{1,0}^2) - \frac{\kappa}{2} P^4 g_0^2 \ln \rho \right) \frac{\rho^2}{f_0^2} + \left(\frac{1}{12} P^2 g_0 (6\alpha_{1,0} P^2 g_0 \kappa + 12k_{1,3,0} - \alpha_{1,0}^3 + 3\alpha_{1,0} \kappa K_0) + \frac{1}{2} P^2 g_0 (4f_{a,3,0} - \alpha_{1,0} P^2 g_0 \kappa) \ln \rho \right) \frac{\rho^3}{f_0^3} + \sum_{n=4}^{\infty} \sum_k k_{1,n,k} \frac{\rho^n}{f_0^n} \ln^k \rho, \quad (5.22)$$

$$K_2 = 1 + \left(-f_{a,3,0} + \frac{3}{2} k_{1,3,0} + 3f_{a,3,0} \ln \rho \right) \frac{\rho^3}{f_0^3} + \sum_{n=4}^{\infty} \sum_k k_{2,n,k} \frac{\rho^n}{f_0^n} \ln^k \rho, \quad (5.23)$$

$$K_3 = K_0 - 2P^2 g_0 \ln \rho - P^2 g_0 \alpha_{1,0} \frac{\rho}{f_0} + \left(\frac{1}{4} P^2 g_0 (\kappa K_0 + 3P^2 g_0 \kappa - \alpha_{1,0}^2) - \frac{\kappa}{2} P^4 g_0^2 \ln \rho \right) \frac{\rho^2}{f_0^2} + \left(\frac{1}{12} P^2 g_0 (6\alpha_{1,0} P^2 g_0 \kappa - 12k_{1,3,0} - \alpha_{1,0}^3 + 3\alpha_{1,0} \kappa K_0) + \frac{1}{2} P^2 g_0 (-4f_{a,3,0} - \alpha_{1,0} P^2 g_0 \kappa) \ln \rho \right) \frac{\rho^3}{f_0^3} + \sum_{n=4}^{\infty} \sum_k k_{3,n,k} \frac{\rho^n}{f_0^n} \ln^k \rho, \quad (5.24)$$

$$g = g_0 \left(1 - \frac{\kappa}{4} P^2 g_0 \frac{\rho^2}{f_0^2} - \frac{\kappa}{4} \alpha_{1,0} P^2 g_0 \frac{\rho^3}{f_0^3} + \sum_{n=4}^{\infty} \sum_k g_{n,k} \frac{\rho^n}{f_0^n} \ln^k \rho \right). \quad (5.25)$$

It is characterized by 12 parameters:

$$\{K_0, f_0, g_0, \alpha_{1,0}, k_{1,3,0}, f_{c,4,0}, f_{a,3,0}, f_{a,4,0}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0}, g_{4,0}\}. \quad (5.26)$$

In what follows we developed the UV expansion to order $\mathcal{O}(\rho^{10})$ inclusive.

5.1.3 IR asymptotics

As in section 3.3, we use a radial coordinate ρ that extends to infinity, see (3.4). The crucial difference between the IR boundary conditions for a chirally symmetric phase discussed in section 3.3 and the IR boundary conditions for a χ SB phase discussed here is that in the former case the manifold \mathcal{M}_5 geodesically completes with (a smooth) shrinking to zero size of $S^3 \subset \mathcal{M}_5$, while in the latter case, much like in supersymmetric Klebanov-Strassler state of cascading gauge theory [2], the 10-dimensional uplift of \mathcal{M}_5 ,

$$\mathcal{M}_5 \rightarrow \mathcal{M}_{10} = \mathcal{M}_5 \times X_5, \quad (5.27)$$

(with the metric given by (2.13)), geodesically completes with (a smooth) shrinking of a 2-cycle in the compact manifold X_5 [2]. Introducing

$$y \equiv \frac{1}{\rho}, \quad h^h \equiv y^{-4} h, \quad f_{a,c}^h \equiv y^2 f_{a,c}, \quad (5.28)$$

the general IR (as $y \rightarrow 0$) asymptotic solution of (5.6)-(5.15) describing the χ SB phase of cascading gauge theory takes form

$$f_1 = f_{1,0}^h + \frac{h_0^h \kappa}{3 f_{1,0}^h} y^2 + \sum_{n=2}^{\infty} f_{1,n}^h y^{2n}, \quad (5.29)$$

$$\begin{aligned} f_c^h = & \frac{3}{4} f_{a,0}^h + \left(-\frac{3 f_{a,0}^h k_{2,4}^h}{2 k_{2,2}^h} + \frac{f_{a,0}^h (k_{1,3}^h)^2}{64 h_0^h P^2 g_0^h} - \frac{13 P^2 g_0^h}{15 (f_{a,0}^h)^2 h_0^h} + \frac{19 (k_{3,1}^h)^2}{320 f_{a,0}^h h_0^h P^2 g_0^h} \right. \\ & \left. - \frac{19 (k_{2,2}^h)^2 P^2 g_0^h}{540 h_0^h} + \frac{6}{5} - \frac{f_{a,0}^h h_0^h \kappa}{2 (f_{1,0}^h)^2} - \frac{27}{5 k_{2,2}^h f_{a,0}^h} + \frac{3 k_{3,1}^h k_{1,3}^h}{20 k_{2,2}^h f_{a,0}^h h_0^h P^2 g_0^h} \right) y^2 \\ & + \sum_{n=2}^{\infty} f_{c,n}^h y^{2n}, \end{aligned} \quad (5.30)$$

$$\begin{aligned} f_a^h = & f_{a,0}^h + \left(\frac{f_{a,0}^h (k_{1,3}^h)^2}{48 h_0^h P^2 g_0^h} - \frac{4 P^2 g_0^h}{45 (f_{a,0}^h)^2 h_0^h} - \frac{17 (k_{3,1}^h)^2}{240 f_{a,0}^h h_0^h P^2 g_0^h} + \frac{17 (k_{2,2}^h)^2 P^2 g_0^h}{405 h_0^h} + \frac{11}{5} \right. \\ & \left. + \frac{f_{a,0}^h k_{2,4}^h}{k_{2,2}^h} + \frac{f_{a,0}^h h_0^h \kappa}{3 (f_{1,0}^h)^2} + \frac{18}{5 k_{2,2}^h f_{a,0}^h} - \frac{k_{3,1}^h k_{1,3}^h}{10 k_{2,2}^h f_{a,0}^h h_0^h P^2 g_0^h} \right) y^2 + \sum_{n=2}^{\infty} f_{a,n}^h y^{2n}, \end{aligned} \quad (5.31)$$

$$f_b = 3 + \left(-\frac{(k_{1,3}^h)^2}{16P^2g_0^h h_0^h} + \frac{4P^2g_0^h}{3(f_{a,0}^h)^3 h_0^h} + \frac{(k_{3,1}^h)^2}{16(f_{a,0}^h)^2 P^2g_0^h h_0^h} - \frac{P^2g_0^h (k_{2,2}^h)^2}{27f_{a,0}^h h_0^h} - \frac{h_0^h \kappa}{(f_{1,0}^h)^2} - \frac{3}{f_{a,0}^h} \right) y^2 + \sum_{n=2}^{\infty} f_{b,n}^h y^{2n}, \quad (5.32)$$

$$h^h = h_0^h + \left(-\frac{(k_{1,3}^h)^2}{48P^2g_0^h} - \frac{4P^2g_0^h}{9(f_{a,0}^h)^3} - \frac{(k_{3,1}^h)^2}{16(f_{a,0}^h)^2 P^2g_0^h} - \frac{P^2g_0^h (k_{2,2}^h)^2}{27f_{a,0}^h} \right) y^2 + \sum_{n=2}^{\infty} h_n^h y^{2n}, \quad (5.33)$$

$$K_1 = k_{1,3}^h y^3 + \left(-\frac{P^2g_0^h k_{1,3}^h (k_{2,2}^h)^2}{54f_{a,0}^h h_0^h} - \frac{9(k_{1,3}^h)^3}{160P^2g_0^h h_0^h} + \frac{6P^2g_0^h k_{1,3}^h}{5(f_{a,0}^h)^3 h_0^h} - \frac{7k_{1,3}^h (k_{3,1}^h)^2}{160(f_{a,0}^h)^2 P^2g_0^h h_0^h} - \frac{4h_0^h k_{1,3}^h \kappa}{5(f_{1,0}^h)^2} - \frac{12k_{1,3}^h}{5f_{a,0}^h} - \frac{9k_{3,1}^h}{5(f_{a,0}^h)^2} + \frac{4P^2g_0^h k_{2,2}^h k_{3,1}^h}{15(f_{a,0}^h)^3 h_0^h} \right) y^5 + \sum_{n=2}^{\infty} k_{1,n}^h y^{2n+1}, \quad (5.34)$$

$$K_2 = k_{2,2}^h y^2 + k_{2,4}^h y^4 + \sum_{n=3}^{\infty} k_{2,n}^h y^{2n}, \quad (5.35)$$

$$K_3 = k_{3,1}^h y + \left(\frac{18k_{3,1}^h}{5(f_{a,0}^h)^2 k_{2,2}^h} + \frac{k_{3,1}^h (k_{1,3}^h)^2}{480h_0^h P^2g_0^h} + \frac{41P^2g_0^h (k_{2,2}^h)^2 k_{3,1}^h}{810f_{a,0}^h h_0^h} + \frac{4P^2g_0^h k_{2,2}^h k_{1,3}^h}{135f_{a,0}^h h_0^h} + \frac{k_{3,1}^h k_{2,4}^h}{k_{2,2}^h} - \frac{k_{1,3}^h (k_{3,1}^h)^2}{10(f_{a,0}^h)^2 h_0^h P^2g_0^h k_{2,2}^h} + \frac{2P^2g_0^h k_{3,1}^h}{15(f_{a,0}^h)^3 h_0^h} - \frac{41(k_{3,1}^h)^3}{480(f_{a,0}^h)^2 h_0^h P^2g_0^h} + \frac{4k_{3,1}^h}{5f_{a,0}^h} + \frac{4h_0^h k_{3,1}^h \kappa}{15(f_{1,0}^h)^2} - \frac{1}{5} k_{1,3}^h \right) y^3 + \sum_{n=2}^{\infty} k_{3,n}^h y^{2n+1}, \quad (5.36)$$

$$g = g_0^h \left(1 + \left(-\frac{(k_{3,1}^h)^2}{16(f_{a,0}^h)^2 h_0^h P^2g_0^h} + \frac{P^2g_0^h (k_{2,2}^h)^2}{27f_{a,0}^h h_0^h} - \frac{(k_{1,3}^h)^2}{48h_0^h P^2g_0^h} + \frac{4P^2g_0^h}{9(f_{a,0}^h)^3 h_0^h} \right) y^2 + \sum_{n=2}^{\infty} g_n^h y^{2n} \right). \quad (5.37)$$

Notice that the prescribed IR boundary conditions imply

$$\lim_{y \rightarrow 0} \Omega_3^2 = \lim_{y \rightarrow 0} \frac{1}{6} f_b h^{1/2} = \lim_{y \rightarrow 0} \frac{y^2}{6} f_b (h^h)^{1/2} = 0, \quad (5.38)$$

with all the other warp factors in (2.13) being finite. Moreover, see (2.13),

$$\lim_{y \rightarrow 0} \left(\Omega_1^2 g_5^2 + \Omega_2^2 [g_3^2 + g_4^2] \right) = \frac{1}{6} f_{a,0}^h (h_0^h)^{1/2} \left(\frac{1}{2} g_5^2 + g_3^2 + g_4^2 \right), \quad (5.39)$$

which is the metric of the round S^3 which stays of finite size in the deep infrared as the 2-cycle fibered over it (smoothly) shrinks to zero size (5.38). Asymptotic solution

(5.29)-(5.37) is characterized by 8 parameters:

$$\{f_{1,0}^h, f_{a,0}^h, h_0^h, k_{1,3}^h, k_{2,2}^h, k_{2,4}^h, k_{3,1}^h, g_0^h\}. \quad (5.40)$$

In what follows we developed the IR expansion to order $\mathcal{O}(y^{10})$ inclusive.

5.1.4 Symmetries and numerical procedure

The background geometry (5.5) dual to a phase of cascading gauge theory with spontaneously broken chiral symmetry on $\mathcal{M}_3^{(\kappa)}$ enjoys all the symmetries²⁴, properly generalized, discussed in section 3.4:

■

$$P \rightarrow \lambda P, \quad g \rightarrow \frac{1}{\lambda} g, \quad \{\rho, f_{1,a,b,c}, h, K_{1,2,3}\} \rightarrow \{\rho, f_{1,a,b,c}, h, K_{1,2,3}\}, \quad (5.41)$$

■

$$P \rightarrow \lambda P, \quad \rho \rightarrow \frac{1}{\lambda} \rho, \quad \{h, K_{1,3}\} \rightarrow \lambda^2 \{h, K_{1,3}\}, \quad \{f_{1,a,b,c}, K_2, g\} \rightarrow \{f_{1,a,b,c}, K_2, g\}, \quad (5.42)$$

■

$$\rho \rightarrow \lambda \rho, \quad f_1 \rightarrow \lambda f_1, \quad \{P, f_{a,b,c}, h, K_{1,2,3}, g\} \rightarrow \{P, f_{a,b,c}, h, K_{1,2,3}, g\}, \quad (5.43)$$

■

$$\begin{pmatrix} P \\ \rho \\ h \\ f_1 \\ f_{a,b,c} \\ K_{1,2,3} \\ g \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{P} \\ \hat{\rho} \\ \hat{h} \\ \hat{f}_1 \\ \hat{f}_{a,b,c} \\ \hat{K}_{1,2,3} \\ \hat{g} \end{pmatrix} = \begin{pmatrix} P \\ \rho/(1+\alpha\rho) \\ (1+\alpha\rho)^4 h \\ f_1 \\ (1+\alpha\rho)^{-2} f_{a,b,c} \\ K_{1,2,3} \\ g \end{pmatrix}, \quad \alpha = \text{const}. \quad (5.44)$$

Thus, much like in section 3.4, we can set

$$g_0 = 1, \quad f_0 = 1, \quad \frac{K_0}{P^2} = \ln \frac{\mu^2}{\Lambda^2 P^2} \equiv \frac{1}{\delta}, \quad (5.45)$$

²⁴We assume that $\kappa \in (0, 1]$.

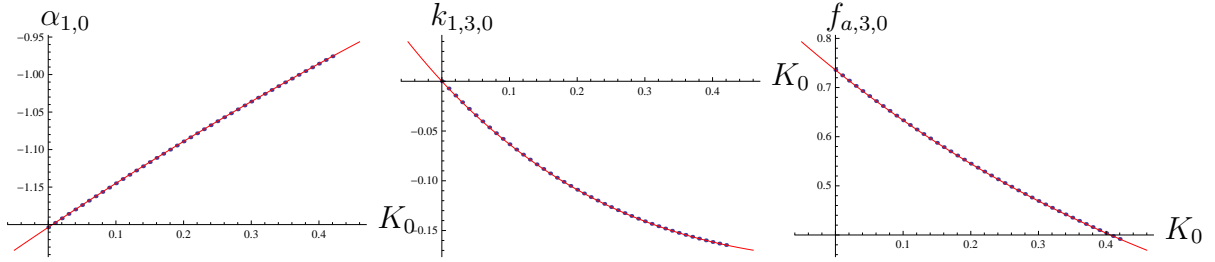


Figure 7: (Colour online) Comparison of values of select UV parameters $\{\alpha_{1,0}, k_{1,3,0}, f_{a,3,0}\}$ of Klebanov-Strassler state obtained numerically (blue dots) with the analytic prediction (red curves), see (5.57).

where $\mu \equiv \frac{1}{f_0}$ is the compactification scale. The residual diffeomorphisms (5.44) are actually completely fixed once we insist on the IR asymptotics as in (5.29)-(5.37).

The numerical procedure for solving the background equations (5.6)-(5.15), subject to the boundary conditions (5.17)-(5.25) and (5.29)-(5.37) is identical to the one described earlier, see section 3.6. Given (5.45), for a fixed δ , the gravitational solution is characterized by 9 parameters in the UV and 8 parameters in the IR:

$$\begin{aligned} \text{UV :} & \quad \{\alpha_{1,0}, k_{1,3,0}, f_{c,4,0}, f_{a,3,0}, f_{a,4,0}, f_{a,6,0}, f_{a,7,0}, f_{a,8,0}, g_{4,0}\}, \\ \text{IR :} & \quad \{f_{1,0}^h, f_{a,0}^h, h_0^h, k_{1,3}^h, k_{2,2}^h, k_{2,4}^h, k_{3,1}^h, g_0^h\}. \end{aligned} \quad (5.46)$$

Notice that $9+8=17$ is precisely the number of integration constants needed to specify a solution to (5.6)-(5.15) — we have 9 second order differential equations and a single first order differential constraint: $2 \times 9 - 1 = 17$.

In practice, we replace the second-order differential equation for f_c (5.7) with the constraint equation (5.15), which we use to algebraically eliminate f'_c from (5.6), (5.8)-(5.14). The solution is found using the “shooting” method as detailed in [15].

Ultimately, we are interested in the solution at $\kappa = 1$. Finding such a “shooting” solution in 17-dimensional parameter space (5.46) is quite challenging. Thus, we start with analytic result for $\kappa = 0$ (the Klebanov-Strassler state of cascading gauge theory), and a fixed value of δ , and slowly increase κ , *i.e.*, continuously deform $\mathcal{M}_3^{(\kappa)}$ from R^3 to S^3 . We further use the obtained solution as a starting point to explore other values of δ .

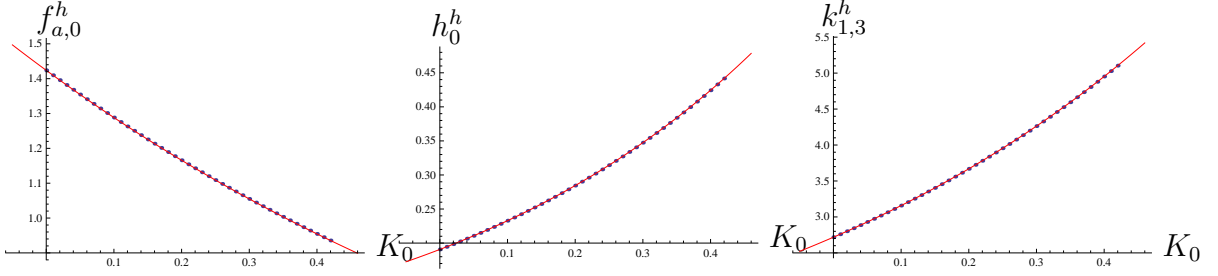


Figure 8: (Colour online) Comparison of values of select IR parameters $\{f_{a,0}^h, h_0^h, k_{1,3}^h\}$ of Klebanov-Strassler state obtained numerically (blue dots) with the analytic prediction (red curves), see (5.58).

5.1.5 κ -deformation of Klebanov-Strassler state

We begin with mapping the Klebanov-Strassler solution [2] to a $\kappa = 0$ solution of (5.6)-(5.15). We set

$$g_0 = 1, \quad P = 1. \quad (5.47)$$

$\mathcal{N} = 1$ supersymmetric Klebanov-Strassler solution takes form²⁵:

$$ds_5^2 = H_{KS}^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H_{KS}^{1/2} \omega_{1,KS}^2 dr^2, \quad (5.48)$$

$$\Omega_i = \omega_{i,KS} H_{KS}^{1/2}, \quad h_i = h_{i,KS},$$

$$h_{1,KS} = \frac{\cosh r - 1}{18 \sinh r} \left(\frac{r \cosh r}{\sinh r} - 1 \right), \quad h_{2,KS} = \frac{1}{18} \left(1 - \frac{r}{\sinh r} \right),$$

$$h_{3,KS} = \frac{\cosh r + 1}{18 \sinh r} \left(\frac{r \cosh r}{\sinh r} - 1 \right), \quad g = 1, \quad f_1 = 1, \quad (5.49)$$

$$\omega_{1,KS} = \frac{\epsilon^{2/3}}{\sqrt{6} \hat{K}_{KS}}, \quad \omega_{2,KS} = \frac{\epsilon^{2/3} \hat{K}_{KS}^{1/2}}{\sqrt{2}} \cosh \frac{r}{2}, \quad \omega_{3,KS} = \frac{\epsilon^{2/3} \hat{K}_{KS}^{1/2}}{\sqrt{2}} \sinh \frac{r}{2},$$

with

$$\hat{K}_{KS} = \frac{(\sinh(2r) - 2r)^{1/3}}{2^{1/3} \sinh r}, \quad H'_{KS} = \frac{16((9h_{2,KS} - 1)h_{1,KS} - 9h_{3,KS}h_{2,KS})}{9\epsilon^{8/3} \hat{K}_{KS}^2 \sinh^2 r}, \quad \Omega_0 = 0, \quad (5.50)$$

where now $r \rightarrow \infty$ is the boundary and $r \rightarrow 0$ is the IR. Above solution is parametrized by a single constant ϵ which will be mapped to K_0 , and which in turn will determine all the parameters in (5.46) once $\kappa = 0$.

²⁵See eqs. (2.22) and (2.34) in [17].

Comparing the metric ansatz in (5.48) and (5.1), (5.5) we identify

$$\frac{(d\rho)^2}{\rho^4} = (w_{1,KS}(r))^2 (dr)^2. \quad (5.51)$$

Introducing

$$z \equiv e^{-r/3}, \quad (5.52)$$

we find from (5.51)

$$\frac{1}{\rho} = \frac{\sqrt{6} (2\epsilon)^{2/3}}{4} \int_1^z du \frac{u^6 - 1}{u^2(1 - u^{12} + 12u^6 \ln u)^{1/3}}. \quad (5.53)$$

In the UV, $r \rightarrow \infty$, $z \rightarrow 0$ and $\rho \rightarrow 0$ we have

$$\begin{aligned} e^{-r/3} \equiv z = & \frac{\sqrt{6} (2\epsilon)^{2/3}}{4} \rho \left(1 + \mathcal{Q}\rho + \mathcal{Q}^2\rho^2 + \mathcal{Q}^3\rho^3 + \mathcal{Q}^4\rho^4 + \mathcal{Q}^5\rho^5 + \left(\frac{27}{80}\epsilon^4 \ln 3 + \mathcal{Q}^6 \right. \right. \\ & + \frac{27}{800}\epsilon^4 - \frac{9}{16}\epsilon^4 \ln 2 + \frac{9}{20}\epsilon^4 \ln \epsilon + \frac{27}{40}\epsilon^4 \ln \rho \Big) \rho^6 + \left(-\frac{63}{16}\epsilon^4 \mathcal{Q} \ln 2 + \frac{189}{80}\epsilon^4 \mathcal{Q} \ln 3 + \mathcal{Q}^7 \right. \\ & + \frac{729}{800}\mathcal{Q}\epsilon^4 + \frac{63}{20}\epsilon^4 \mathcal{Q} \ln \epsilon + \frac{189}{40}\mathcal{Q}\epsilon^4 \ln \rho \Big) \rho^7 + \left(\frac{2403}{400}\epsilon^4 \mathcal{Q}^2 - \frac{63}{4}\epsilon^4 \mathcal{Q}^2 \ln 2 + \frac{189}{20}\epsilon^4 \mathcal{Q}^2 \ln 3 \right. \\ & + \frac{63}{5}\epsilon^4 \mathcal{Q}^2 \ln \epsilon + \mathcal{Q}^8 + \frac{189}{10}\epsilon^4 \mathcal{Q}^2 \ln \rho \Big) \rho^8 + \left(\frac{189}{5}\epsilon^4 \mathcal{Q}^3 \ln \epsilon + \frac{9729}{400}\epsilon^4 \mathcal{Q}^3 - \frac{189}{4}\epsilon^4 \mathcal{Q}^3 \ln 2 \right. \\ & \left. \left. + \frac{567}{20}\epsilon^4 \mathcal{Q}^3 \ln 3 + \mathcal{Q}^9 + \frac{567}{10}\epsilon^4 \mathcal{Q}^3 \ln \rho \right) \rho^9 + \mathcal{O}(\rho^{10} \ln \rho) \right), \end{aligned} \quad (5.54)$$

where

$$\begin{aligned} \mathcal{Q} = & \frac{\sqrt{6} (2\epsilon)^{2/3}}{4} \left\{ \int_0^1 du \left(\frac{1 - u^6}{u^2(1 - u^{12} + 12u^6 \ln u)^{1/3}} - \frac{1}{u^2} \right) - 1 \right\} \\ = & -\frac{\sqrt{6} (2\epsilon)^{2/3}}{4} \times 0.839917(9). \end{aligned} \quad (5.55)$$

In the IR, $r \rightarrow 0$, $z \rightarrow 1_-$ and $\frac{1}{\rho} \rightarrow 0$ we have

$$r = \frac{\sqrt{6} 2^{1/3}}{3^{1/3} \epsilon^{2/3}} y \left(1 - \frac{2^{2/3} 3^{1/3}}{15 \epsilon^{4/3}} y^2 + \frac{71 3^{2/3} 2^{1/3}}{2625 \epsilon^{8/3}} y^4 + \mathcal{O}(y^6) \right). \quad (5.56)$$

Using (5.54) and (5.56), and the exact analytic solution describing the Klebanov-Strassler state of cascading gauge theory (5.49), (5.50) we can identify parameters²⁶

²⁶We matched the asymptotic expansions (5.17)-(5.25) and (5.29)-(5.37) with the exact solution (5.49) to the order we developed them: $\mathcal{O}(\rho^{10})$ and $\mathcal{O}(y^{10})$ correspondingly.

(5.46)

$$\begin{aligned}
K_0 &= -\ln 3 + \frac{5}{3} \ln 2 - \frac{4}{3} \ln \epsilon - \frac{2}{3}, \\
a_{1,0} &= 2\mathcal{Q}, \quad k_{1,3,0} = \frac{\epsilon^2 \sqrt{6}}{4} (-5 \ln 2 + 3 \ln 3 + 4 \ln \epsilon + 2), \quad f_{c,4,0} = 0, \\
f_{a,3,0} &= \frac{3\sqrt{6}}{4} \epsilon^2, \quad f_{a,4,0} = \frac{3\sqrt{6}}{4} \epsilon^2 \mathcal{Q} \\
f_{a,6,0} &= \frac{3\epsilon^2}{400} (-225\epsilon^2 \ln 2 + 180\epsilon^2 \ln \epsilon + 216\epsilon^2 + 135\epsilon^2 \ln 3 + 100\sqrt{6}\mathcal{Q}^3), \\
f_{a,7,0} &= \frac{3\sqrt{6}}{4} \epsilon^2 \mathcal{Q}^4, \\
f_{a,8,0} &= \frac{3\epsilon^2}{16} \mathcal{Q}^2 (4\sqrt{6}\mathcal{Q}^3 + 135\epsilon^2 - 90\epsilon^2 \ln 2 + 54\epsilon^2 \ln 3 + 72\epsilon^2 \ln \epsilon), \quad g_{4,0} = 0,
\end{aligned} \tag{5.57}$$

in the UV, and

$$\begin{aligned}
f_{1,0}^h &= 1, \quad f_{a,0}^h = 2^{1/3} 3^{2/3} \epsilon^{4/3}, \quad h_0^h = \epsilon^{-8/3} \times 0.056288(0), \\
k_{1,3}^h &= \frac{4\sqrt{6}}{9 \epsilon^2}, \quad k_{2,2}^h = \frac{2^{2/3}}{3^{2/3} \epsilon^{4/3}}, \quad k_{2,4}^h = -\frac{11}{45} \frac{2^{1/3} 3^{2/3}}{\epsilon^{8/3}}, \\
k_{3,1}^h &= \frac{4\sqrt{6} 2^{1/3} 3^{2/3}}{27 \epsilon^{2/3}}, \quad g_0^h = 1,
\end{aligned} \tag{5.58}$$

in the IR. Notice that inverting the first identification in (5.57), $\epsilon = \epsilon(K_0)$, we obtain a prediction for all the parameters (5.46) as a function of K_0 .

Figures 7 and 8 compares the results of select UV and IR parameters in (5.46) obtained numerically (blue dots) with analytic predictions (red curves) (5.57) and (5.58) for the supersymmetric Klebanov-Strassler state. In this numerical computation we must set $\kappa = 0$, $f_1(\rho) \equiv 1$, *i.e.*, we remove the differential equation (5.6). Correspondingly, we have to remove (fix) 2 parameters in (5.46) for the numerical shooting procedure to be well-posed. Requiring that $f_1 \equiv 1$ (for $\kappa = 0$) both in the UV asymptotic solution (5.17) and the IR asymptotic solution (5.29) implies

$$f_{a,4,0} = \frac{1}{2} \alpha_{1,0} f_{a,3,0}, \quad f_{1,0}^h = 1. \tag{5.59}$$

Notice that in Klebanov-Strassler state the string coupling is identically constant, *i.e.*, $g = 1$. The latter in particular implies that $g_{4,0} = 0$ and $g_0^h = 1$. Numerically, over the range of values K_0 in figure 7, we find

$$g_{4,0} \sim 10^{-6} \dots 10^{-5}, \quad |1 - g_0^h| \sim 10^{-9} \dots 10^{-8}. \tag{5.60}$$

As we mentioned earlier, we are after the states of cascading gauge theory with broken chiral symmetry on S^3 , *i.e.*, the deformations of Klebanov-Strassler states at $\kappa = 1$. In practice we start with numerical Klebanov-Strassler state at $K_0 = 0.25$ ($P = 1$) and increase κ in increments of $\delta\kappa = 10^{-3}$ up to $\kappa = 1$. The resulting state is then used as a starting point to explore the states of cascading gauge theory on S^3 with χ SB for other values of $K_0 \neq 0.25$.

5.2 The first-order χ SB phase transition in S^3 -compactified cascading gauge theory

In section 5.1.5 we numerically constructed states of cascading gauge theory on S^3 with spontaneously broken chiral symmetry over a range of $\ln \frac{\mu}{\Lambda}$. To determine whether (and when in terms of $\ln \frac{\mu}{\Lambda}$) these states represent the true ground state²⁷ of S^3 -compactified cascading gauge theory one has to compute their energies. The energy density of a chirally symmetric state was computed in (3.90) using the full holographic renormalization of cascading gauge theory implemented in [11]. To compute the energy density of the state of cascading gauge theory with spontaneously broken chiral symmetry one has to properly refine the holographic renormalization of [11]. We explain the main features of such refinement here.

For a static S^3 -invariant states described by the effective action (2.1) the energy density is given

$$\mathcal{E} = \int_{\rho_{UV}}^{\infty} d\rho \mathcal{L}_E, \quad (5.61)$$

where \mathcal{L}_E is the Euclidean one-dimensional Lagrangian density corresponding to the state, and ρ_{UV} is the UV cut-off, regularizing the Euclidean gravitational action in (5.61). Briefly, holographic renormalization of the theory modifies the energy density

$$\int_{\rho_{UV}}^{\infty} d\rho \mathcal{L}_E \rightarrow \int_{\rho_{UV}}^{\infty} d\rho \mathcal{L}_E + S_{GH}^{\rho_{UV}} + S_{counterterms}^{\rho_{UV}}, \quad (5.62)$$

to include the Gibbons-Hawking and the local counterterms at the cut-off boundary $\rho = \rho_{UV}$ in a way that would render the renormalized energy density finite in the limit $\rho_{UV} \rightarrow 0$.

Using the equations of motion (5.6)-(5.15), it is possible to show that the on-shell gravitational action (2.1) for static, S^3 -invariant states of cascading gauge theory is a

²⁷As opposite to the states of cascading gauge theory on S^3 with unbroken chiral symmetry discussed in section 3.

total derivative. Specifically, we find²⁸

$$\mathcal{L}_E^b = \frac{108}{16\pi G_5} \times \frac{d}{d\rho} \left(\frac{2c_2^3 c_1' \Omega_1 \Omega_2^2 \Omega_3^2}{c_3} \right) = -\frac{108}{16\pi G_5} \times \frac{d}{d\rho} \left(\frac{f_1^3 f_c^{1/2} f_a f_b (\rho h' + 4h)}{216h\rho^4} \right). \quad (5.63)$$

The integral in (5.61) now becomes the boundary values of the expression in (5.63). Note that

$$\lim_{\rho \rightarrow \infty} \frac{f_1^3 f_c^{1/2} f_a f_b (\rho h' + 4h)}{216h\rho^4} = -\lim_{y \rightarrow 0} \frac{f_1^3 (f_c^h)^{1/2} f_a^h f_b y^2 (h^h)'}{216h^h} = 0, \quad (5.64)$$

where in the last equality we used (5.29)-(5.33). Thus,

$$\frac{16\pi G_5}{108} \mathcal{E}^b = \left\{ \mathcal{E}_{-4}^b \frac{1}{\rho^4} + \mathcal{E}_{-3}^b \frac{1}{\rho^3} + \mathcal{E}_{-2}^b \frac{1}{\rho^2} + \mathcal{E}_{-1}^b \frac{1}{\rho} + \mathcal{E}_0^b + \mathcal{O}(\rho) \right\} \Big|_{\rho=\rho_{UV}}, \quad (5.65)$$

with

$$\mathcal{E}_{-4}^b = \frac{K_0 - 2 \ln \rho}{27(2K_0 + 1 - 4 \ln \rho)}, \quad (5.66)$$

$$\mathcal{E}_{-3}^b = -\frac{\alpha_{1,0}^b}{27(2K_0 + 1 - 4 \ln \rho)^2} \left(2K_0 + 4K_0^2 + 1 - (16K_0 + 4) \ln \rho + 16 \ln^2 \rho \right), \quad (5.67)$$

$$\begin{aligned} \mathcal{E}_{-2}^b = & -\frac{1}{3888(2K_0 + 1 - 4 \ln \rho)^3} \left(-720(\alpha_{1,0}^b)^2 K_0 - 117 - 864(\alpha_{1,0}^b)^2 K_0^3 - 394K_0 \right. \\ & - 312K_0^3 + 36(\alpha_{1,0}^b)^2 - 864(\alpha_{1,0}^b)^2 K_0^2 - 476K_0^2 + (3456(\alpha_{1,0}^b)^2 K_0 + 788 \\ & + 1440(\alpha_{1,0}^b)^2 + 1904K_0 + 5184(\alpha_{1,0}^b)^2 K_0^2 + 1872K_0^2) \ln \rho + (-3744K_0 \\ & \left. - 3456(\alpha_{1,0}^b)^2 - 1904 - 10368(\alpha_{1,0}^b)^2 K_0) \ln^2 \rho + (2496 + 6912(\alpha_{1,0}^b)^2) \ln^3 \rho \right), \end{aligned} \quad (5.68)$$

$$\begin{aligned} \mathcal{E}_{-1}^b = & -\frac{1}{3888(2K_0 + 1 - 4 \ln \rho)^4} \alpha_{1,0}^b \left(-191 + 300(\alpha_{1,0}^b)^2 + 1248K_0^2 + 1264K_0^3 \right. \\ & + 624K_0^4 + 4K_0 + 864(\alpha_{1,0}^b)^2 K_0^3 + 1056(\alpha_{1,0}^b)^2 K_0^2 + 576(\alpha_{1,0}^b)^2 K_0^4 - 168(\alpha_{1,0}^b)^2 K_0 \\ & (336(\alpha_{1,0}^b)^2 - 8 - 4992K_0 - 4608(\alpha_{1,0}^b)^2 K_0^3 - 5184(\alpha_{1,0}^b)^2 K_0^2 - 4992K_0^3 \\ & - 4224(\alpha_{1,0}^b)^2 K_0 - 7584K_0^2) \ln \rho + (4992 + 4224(\alpha_{1,0}^b)^2 + 15168K_0 \\ & + 13824(\alpha_{1,0}^b)^2 K_0^2 + 14976K_0^2 + 10368(\alpha_{1,0}^b)^2 K_0) \ln^2 \rho + (-6912(\alpha_{1,0}^b)^2 \\ & \left. - 18432(\alpha_{1,0}^b)^2 K_0 - 19968K_0 - 10112) \ln^3 \rho + (9216(\alpha_{1,0}^b)^2 + 9984) \ln^4 \rho \right), \end{aligned} \quad (5.69)$$

²⁸See (5.1) and (5.5) for the background metric.

$$\begin{aligned}\mathcal{E}_0^b = & -\frac{1}{648}\ln^2\rho + \left(\frac{5}{2592} + \frac{1}{648}K_0\right)\ln\rho + \frac{23}{9216} + \frac{1}{5184}K_0 + \frac{1}{864}(\alpha_{1,0}^b)^4 \\ & + \frac{1}{54}\alpha_{1,0}^b f_{a,3,0} + \frac{7}{5184}(\alpha_{1,0}^b)^2 - \frac{1}{27}f_{a,4,0} + \frac{1}{54}f_{c,4,0} + \mathcal{O}(\ln^{-1}\rho),\end{aligned}\quad (5.70)$$

where we set $P = 1$, $g_0 = 1$, $f_0 = 1$, and used (5.17)-(5.21). The superscript b in the UV parameter $\alpha_{1,0}$ is used to indicate that it is computed in the phase with broken chiral symmetry.

Clearly, the expression (5.65) is divergent in the limit $\rho_{UV} \rightarrow 0$. Turns out that all the divergences are removed once we include the generalized²⁹ Gibbons-Hawking term, see [11],

$$S_{GH}^{\rho_{UV}} = \frac{108}{8\pi G_5} \frac{1}{c_3} \left(c_1 c_2^3 \Omega_1 \Omega_2^2 \Omega_3^2 \right)' \Big|_{\rho=\rho_{UV}} = \frac{1}{8\pi G_5} \frac{\rho}{h^{1/4}} \left(\frac{h^{1/4} f_1^3 f_c^{1/2} f_a f_b}{\rho^4} \right)' \Big|_{\rho=\rho_{UV}}, \quad (5.71)$$

and the local counter-terms obtained in [11] with the following obvious modifications:

$$K^{KT} = \frac{1}{2}K_1 + \frac{1}{2}K_3, \quad \Omega_1^{KT} = 3\Omega_1, \quad \Omega_2^{KT} = \frac{\sqrt{6}}{2}(\Omega_2 + \Omega_3). \quad (5.72)$$

We find

$$\mathcal{E}^b = \frac{1}{8\pi G_5} \left(\frac{403}{1920} + \frac{1}{32}K_0^2 + \frac{3}{32}K_0 - 3f_{a,4,0} + \frac{3}{2}f_{c,4,0} + \frac{3}{2}a_{1,0}^b f_{a,3,0} - \frac{3}{32}(\alpha_{1,0}^b)^2 \right). \quad (5.73)$$

Notice that (5.73) coincides with (3.90) once restricted to chirally symmetric states:

$$f_{a,3,0} = 0, \quad f_{c,4,0} = a_{4,0}, \quad f_{a,4,0} = b_{4,0}. \quad (5.74)$$

We can now compare the energy densities of a chirally symmetric state and a state spontaneously breaking chiral symmetry for cascading gauge theory on S^3 (we restored the full $\{P, g_0, f_0\}$ dependence)

$$\begin{aligned}\mathcal{E}^b - \mathcal{E}^s = & \frac{1}{8\pi G_5} \frac{1}{f_0^4} \left(3(b_{4,0} - f_{a,4,0}) + \frac{3}{2}(f_{c,4,0} - a_{4,0}) + \frac{3}{2}a_{1,0}^b f_{a,3,0} \right. \\ & \left. + \frac{3}{32}((\alpha_{1,0}^s)^2 - (\alpha_{1,0}^b)^2) \right).\end{aligned}\quad (5.75)$$

²⁹“Generalized” five-dimensional Gibbons-Hawking term is just a dimensional reduction of the 10-dimensional Gibbons-Hawking term corresponding to (2.13).

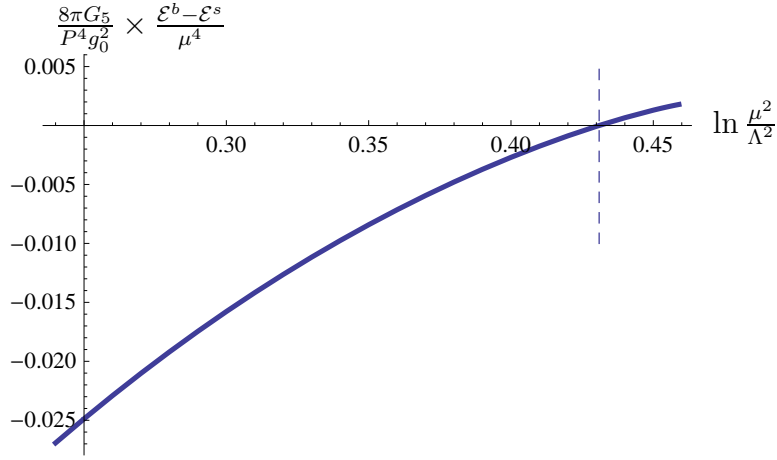


Figure 9: Energy densities difference between the state with spontaneously broken chiral symmetry, \mathcal{E}^b , and the chirally symmetric state, \mathcal{E}^s , of cascading gauge theory on S^3 as a function of compactification scale μ and the strong coupling scale Λ of the theory. The vertical dashed line, see (5.76), indicates the location of the first-order chiral symmetry breaking QPT in cascading gauge theory.

Figure 9 presents the energy densities difference between the state with spontaneously broken chiral symmetry, \mathcal{E}^b , and the chirally symmetric state, \mathcal{E}^s , of cascading gauge theory on S^3 as a function of $\ln \frac{\mu^2}{\Lambda^2}$. Notice that the for $\mu > \mu_{\chi\text{SB}}$ (indicated by a vertical dashed line),

$$\mu_{\chi\text{SB}} = 1.240467(8) \Lambda, \quad (5.76)$$

$\mathcal{E}^b > \mathcal{E}^s$, *i.e.*, the true ground state of cascading gauge theory on S^3 is chirally symmetric. Since

$$\left. \frac{d(\mathcal{E}^b - \mathcal{E}^s)}{d \ln \mu} \right|_{\mu=\mu_{\chi\text{SB}}} \neq 0, \quad (5.77)$$

at $\mu = \mu_{\chi\text{SB}}$ the theory undergoes the first-order quantum phase transition associated with spontaneous breaking of chiral symmetry. Finally, notice that $\mu_{\chi\text{SB}} > \mu_c$ (see (4.51)) associated with the condensation of χSB tachyons in a chirally symmetric phase of the theory.

Acknowledgments

I would like to thank Ofer Aharony, Philip Argyres, Cliff Burgess, Rob Myers and Andrei Starinets for valuable discussions. I would like to thank Aspen Center for Physics,

Galileo Galilei Institute for Theoretical Physics, International Centre for Mathematical Sciences and Centro de Ciencias de Benasque Pedro Pascual for hospitality during the various stages of this project. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. I gratefully acknowledge further support by an NSERC Discovery grant.

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